Problem A.20

Show that the first, second, and last coefficients in the characteristic equation (Equation A.73) are:

$$C_n = (-1)^n \ , \ C_{n-1} = (-1)^{n-1}\,\mathrm{Tr}(\mathsf{T}), \text{ and } C_0 = \det(\mathsf{T}). \tag{A.92}$$

For a 3×3 matrix with elements T_{ij}, what is C_1?

Solution

An $n \times n$ matrix representing a linear transformation $\hat{T}$ has the general form,

$$\mathsf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & \cdots & T_{1n} \\ T_{21} & T_{22} & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{33} & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{43} & T_{44} & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn} \end{bmatrix},$$

in a basis. The eigenvalue problem it satisfies is

$$\mathsf{Ta} = \lambda \mathsf{a},$$

which implies that

$$\det(\mathsf{T} - \lambda \mathsf{I}) = 0$$

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} & T_{14} & \cdots & T_{1n} \\ T_{21} & T_{22}-\lambda & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} = 0. \tag{1}$$

This determinant is in general an nth-degree polynomial.

$$C_n \lambda^n + C_{n-1} \lambda^{n-1} + \cdots + C_1 \lambda + C_0 = 0$$

To determine C_0, set $\lambda = 0$ in the last two equations.

$$C_0 = \begin{vmatrix} T_{11} & T_{12} & T_{13} & T_{14} & \cdots & T_{1n} \\ T_{21} & T_{22} & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{33} & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{43} & T_{44} & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn} \end{vmatrix} = \det(\mathsf{T})$$

To determine C_n and C_{n-1}, use expansion by minors to evaluate the determinant in equation (1).

$$0 = \begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} & T_{14} & \cdots & T_{1n} \\ T_{21} & T_{22}-\lambda & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} \tag{1}$$

$$= (T_{11}-\lambda)\begin{vmatrix} T_{22}-\lambda & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{32} & T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{42} & T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} - T_{12}\begin{vmatrix} T_{21} & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix}$$

$$+ T_{13}\begin{vmatrix} T_{21} & T_{22}-\lambda & T_{24} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{34} & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} - T_{14}\begin{vmatrix} T_{21} & T_{22}-\lambda & T_{23} & \cdots & T_{2n} \\ T_{31} & T_{32} & T_{33}-\lambda & \cdots & T_{3n} \\ T_{41} & T_{42} & T_{43} & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & \cdots & T_{nn}-\lambda \end{vmatrix} + \cdots$$

Notice that in every term after the first, two factors of λ are wiped out. This means that all of the information about C_n and C_{n-1} is contained in the first term.

$$0 = (T_{11}-\lambda)\begin{vmatrix} T_{22}-\lambda & T_{23} & T_{24} & \cdots & T_{2n} \\ T_{32} & T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{42} & T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n2} & T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} + \cdots$$

$$= (T_{11}-\lambda)\left[(T_{22}-\lambda)\begin{vmatrix} T_{33}-\lambda & T_{34} & \cdots & T_{3n} \\ T_{43} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n3} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} - T_{23}\begin{vmatrix} T_{32} & T_{34} & \cdots & T_{3n} \\ T_{42} & T_{44}-\lambda & \cdots & T_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n2} & T_{n4} & \cdots & T_{nn}-\lambda \end{vmatrix} + \cdots\right] + \cdots$$

Again, in every term after the first, two factors of λ are wiped out.

This means that all of the information about C_n and C_{n-1} is contained in the first term.

$$0 = (T_{11} - \lambda)(T_{22} - \lambda)\begin{vmatrix} T_{33} - \lambda & T_{34} & \cdots & T_{3n} \\ T_{43} & T_{44} - \lambda & \cdots & T_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n3} & T_{n4} & \cdots & T_{nn} - \lambda \end{vmatrix} + \cdots$$

Continuing in this fashion results in

$$0 = (T_{11} - \lambda)(T_{22} - \lambda)(T_{33} - \lambda)\cdots(T_{nn} - \lambda) + \cdots .$$

Now multiply the terms on the right side, looking at the two highest powers of λ specifically.

$$\begin{aligned} 0 &= \overbrace{(-\lambda)(-\lambda)(-\lambda)\cdots(-\lambda)}^{n \text{ times}} + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}} T_{nn} + \cdots \\ &\quad + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}} T_{33} + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}} T_{22} + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}} T_{11} + \cdots \\ &= (-\lambda)^n + (-\lambda)^{n-1}T_{nn} + \cdots + (-\lambda)^{n-1}T_{33} + (-\lambda)^{n-1}T_{22} + (-\lambda)^{n-1}T_{11} + \cdots \\ &= (-\lambda)^n + (-\lambda)^{n-1}(T_{nn} + \cdots + T_{33} + T_{22} + T_{11}) + \cdots \\ &= (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}\,\mathrm{Tr}(\mathsf{T}) + \cdots \\ &= \underbrace{(-1)^n}_{C_n}\lambda^n + \underbrace{(-1)^{n-1}\,\mathrm{Tr}(\mathsf{T})}_{C_{n-1}}\lambda^{n-1} + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} C_n &= (-1)^n \\ C_{n-1} &= (-1)^{n-1}\,\mathrm{Tr}(\mathsf{T}) \\ C_0 &= \det(\mathsf{T}). \end{aligned}$$

For a 3×3 matrix,

$$\mathsf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad \Rightarrow \quad \det(\mathsf{T} - \lambda \mathsf{I}) = 0 \quad \rightarrow \quad \begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0.$$

Evaluate the determinant.

$$0 = \begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix}$$

$$= (T_{11} - \lambda) \begin{vmatrix} T_{22} - \lambda & T_{23} \\ T_{32} & T_{33} - \lambda \end{vmatrix} - T_{12} \begin{vmatrix} T_{21} & T_{23} \\ T_{31} & T_{33} - \lambda \end{vmatrix} + T_{13} \begin{vmatrix} T_{21} & T_{22} - \lambda \\ T_{31} & T_{32} \end{vmatrix}$$

$$= (T_{11} - \lambda)[(T_{22} - \lambda)(T_{33} - \lambda) - T_{23}T_{32}] - T_{12}[T_{21}(T_{33} - \lambda) - T_{23}T_{31}] + T_{13}[T_{21}T_{32} - T_{31}(T_{22} - \lambda)]$$

$$= -\lambda^3 + (T_{11} + T_{22} + T_{33})\lambda^2 + (T_{12}T_{21} - T_{11}T_{22} + T_{13}T_{31} - T_{11}T_{33} + T_{23}T_{32} - T_{22}T_{33})\lambda$$
$$+ (T_{11}T_{22}T_{33} - T_{12}T_{21}T_{33} + T_{13}T_{21}T_{32} - T_{11}T_{23}T_{32} + T_{12}T_{23}T_{31} - T_{13}T_{22}T_{31})$$

C_1 is the coefficient of λ. Therefore,

$$C_1 = T_{12}T_{21} - T_{11}T_{22} + T_{13}T_{31} - T_{11}T_{33} + T_{23}T_{32} - T_{22}T_{33}.$$