Problem A.24

Show that the $\tilde{\mathsf{a}}$ computed from Equations A.88 and A.90 are eigenvectors of V.

Solution

Show that $\tilde{\mathsf{a}}^{(1)}$ is an eigenvector of V first. According to Equation A.88 on page 480, $\tilde{\mathsf{a}}^{(1)} = d_{11}\mathsf{a}^{(1)} + d_{21}\mathsf{a}^{(2)}$.

$$\begin{aligned}\mathsf{V}\tilde{\mathsf{a}}^{(1)} &= \mathsf{V}(d_{11}\mathsf{a}^{(1)} + d_{21}\mathsf{a}^{(2)}) \\ &= d_{11}\mathsf{V}\mathsf{a}^{(1)} + d_{21}\mathsf{V}\mathsf{a}^{(2)} \\ &= d_{11}(c_{11}\mathsf{a}^{(1)} + c_{21}\mathsf{a}^{(2)}) + d_{21}(c_{12}\mathsf{a}^{(1)} + c_{22}\mathsf{a}^{(2)}) \\ &= (c_{11}d_{11} + c_{12}d_{21})\mathsf{a}^{(1)} + (c_{21}d_{11} + c_{22}d_{21})\mathsf{a}^{(2)}\end{aligned}$$

Equation A.90 on page 480 says that $c_{11}d_{11} + c_{12}d_{21} = v_1 d_{11}$ and $c_{21}d_{11} + c_{22}d_{21} = v_1 d_{21}$.

$$\begin{aligned}\mathsf{V}\tilde{\mathsf{a}}^{(1)} &= v_1 d_{11}\mathsf{a}^{(1)} + v_1 d_{21}\mathsf{a}^{(2)} \\ &= v_1(d_{11}\mathsf{a}^{(1)} + d_{21}\mathsf{a}^{(2)}) \\ &= v_1\tilde{\mathsf{a}}^{(1)}\end{aligned}$$

Now show that $\tilde{\mathsf{a}}^{(2)}$ is an eigenvector of V. According to Equation A.88 on page 480, $\tilde{\mathsf{a}}^{(2)} = d_{12}\mathsf{a}^{(1)} + d_{22}\mathsf{a}^{(2)}$.

$$\begin{aligned}\mathsf{V}\tilde{\mathsf{a}}^{(2)} &= \mathsf{V}(d_{12}\mathsf{a}^{(1)} + d_{22}\mathsf{a}^{(2)}) \\ &= d_{12}\mathsf{V}\mathsf{a}^{(1)} + d_{22}\mathsf{V}\mathsf{a}^{(2)} \\ &= d_{12}(c_{11}\mathsf{a}^{(1)} + c_{21}\mathsf{a}^{(2)}) + d_{22}(c_{12}\mathsf{a}^{(1)} + c_{22}\mathsf{a}^{(2)}) \\ &= (c_{11}d_{12} + c_{12}d_{22})\mathsf{a}^{(1)} + (c_{21}d_{12} + c_{22}d_{22})\mathsf{a}^{(2)}\end{aligned}$$

Equation A.90 on page 480 says that $c_{11}d_{12} + c_{12}d_{22} = v_2 d_{12}$ and $c_{21}d_{12} + c_{22}d_{22} = v_2 d_{22}$.

$$\begin{aligned}\mathsf{V}\tilde{\mathsf{a}}^{(2)} &= v_2 d_{12}\mathsf{a}^{(1)} + v_2 d_{22}\mathsf{a}^{(2)} \\ &= v_2(d_{12}\mathsf{a}^{(1)} + d_{22}\mathsf{a}^{(2)}) \\ &= v_2\tilde{\mathsf{a}}^{(2)}\end{aligned}$$