Problem A.27

A hermitian linear transformation must satisfy $\langle\alpha\,|\,\hat{T}\beta\rangle = \langle\hat{T}\alpha\,|\,\beta\rangle$ for all vectors $|\alpha\rangle$ and $|\beta\rangle$. Prove that it is (surprisingly) sufficient that $\langle\gamma\,|\,\hat{T}\gamma\rangle = \langle\hat{T}\gamma\,|\,\gamma\rangle$ for all vectors $|\gamma\rangle$. *Hint:* First let $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$, and then let $|\gamma\rangle = |\alpha\rangle + i|\beta\rangle$.

Solution

The aim is to show that if $\langle\gamma|\hat{T}\gamma\rangle = \langle\hat{T}\gamma|\gamma\rangle$ for all $|\gamma\rangle$, then $\langle\alpha|\hat{T}\beta\rangle = \langle\hat{T}\alpha|\beta\rangle$ for all $|\alpha\rangle$ and $|\beta\rangle$. Suppose that $\langle\gamma|\hat{T}\gamma\rangle = \langle\hat{T}\gamma|\gamma\rangle$ for all $|\gamma\rangle$, and expand the left side assuming that $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$.

$$\begin{aligned}
\langle\gamma|\hat{T}\gamma\rangle &= \langle\gamma\,|\,\hat{T}\,|\,\gamma\rangle \\
&= (\mathsf{a}+\mathsf{b})^\dagger\mathsf{T}(\mathsf{a}+\mathsf{b}) \\
&= (\mathsf{a}^\dagger+\mathsf{b}^\dagger)\mathsf{T}(\mathsf{a}+\mathsf{b}) \\
&= (\mathsf{a}^\dagger+\mathsf{b}^\dagger)(\mathsf{Ta}+\mathsf{Tb}) \\
&= \mathsf{a}^\dagger(\mathsf{Ta}+\mathsf{Tb}) + \mathsf{b}^\dagger(\mathsf{Ta}+\mathsf{Tb}) \\
&= \mathsf{a}^\dagger\mathsf{Ta} + \mathsf{a}^\dagger\mathsf{Tb} + \mathsf{b}^\dagger\mathsf{Ta} + \mathsf{b}^\dagger\mathsf{Tb} \\
&= \langle\alpha\,|\,\hat{T}\,|\,\alpha\rangle + \langle\alpha\,|\,\hat{T}\,|\,\beta\rangle + \langle\beta\,|\,\hat{T}\,|\,\alpha\rangle + \langle\beta\,|\,\hat{T}\,|\,\beta\rangle \\
&= \langle\alpha|\hat{T}\alpha\rangle + \langle\alpha|\hat{T}\beta\rangle + \langle\beta|\hat{T}\alpha\rangle + \langle\beta|\hat{T}\beta\rangle
\end{aligned}$$

Expand the right side now with $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$.

$$\begin{aligned}
\langle\hat{T}\gamma|\gamma\rangle &= \langle\gamma\,|\,\hat{T}^\dagger\,|\,\gamma\rangle \\
&= (\mathsf{a}+\mathsf{b})^\dagger\mathsf{T}^\dagger(\mathsf{a}+\mathsf{b}) \\
&= (\mathsf{a}^\dagger+\mathsf{b}^\dagger)\mathsf{T}^\dagger(\mathsf{a}+\mathsf{b}) \\
&= (\mathsf{a}^\dagger\mathsf{T}^\dagger+\mathsf{b}^\dagger\mathsf{T}^\dagger)(\mathsf{a}+\mathsf{b}) \\
&= \mathsf{a}^\dagger\mathsf{T}^\dagger(\mathsf{a}+\mathsf{b}) + \mathsf{b}^\dagger\mathsf{T}^\dagger(\mathsf{a}+\mathsf{b}) \\
&= \mathsf{a}^\dagger\mathsf{T}^\dagger\mathsf{a} + \mathsf{a}^\dagger\mathsf{T}^\dagger\mathsf{b} + \mathsf{b}^\dagger\mathsf{T}^\dagger\mathsf{a} + \mathsf{b}^\dagger\mathsf{T}^\dagger\mathsf{b} \\
&= \langle\alpha\,|\,\hat{T}^\dagger\,|\,\alpha\rangle + \langle\alpha\,|\,\hat{T}^\dagger\,|\,\beta\rangle + \langle\beta\,|\,\hat{T}^\dagger\,|\,\alpha\rangle + \langle\beta\,|\,\hat{T}^\dagger\,|\,\beta\rangle \\
&= \langle\hat{T}\alpha|\alpha\rangle + \langle\hat{T}\alpha|\beta\rangle + \langle\hat{T}\beta|\alpha\rangle + \langle\hat{T}\beta|\beta\rangle
\end{aligned}$$

Since $\langle\gamma\,|\,\hat{T}\gamma\rangle = \langle\hat{T}\gamma\,|\,\gamma\rangle$ and $\langle\alpha\,|\,\hat{T}\alpha\rangle = \langle\hat{T}\alpha\,|\,\alpha\rangle$ and $\langle\beta\,|\,\hat{T}\beta\rangle = \langle\hat{T}\beta\,|\,\beta\rangle$,

$$\cancel{\langle\alpha|\hat{T}\alpha\rangle} + \langle\alpha|\hat{T}\beta\rangle + \langle\beta|\hat{T}\alpha\rangle + \cancel{\langle\beta|\hat{T}\beta\rangle} = \cancel{\langle\hat{T}\alpha|\alpha\rangle} + \langle\hat{T}\alpha|\beta\rangle + \langle\hat{T}\beta|\alpha\rangle + \cancel{\langle\hat{T}\beta|\beta\rangle}$$

$$\langle\alpha|\hat{T}\beta\rangle + \langle\beta|\hat{T}\alpha\rangle = \langle\hat{T}\alpha|\beta\rangle + \langle\hat{T}\beta|\alpha\rangle. \tag{1}$$

Expand the left side now with $|\gamma\rangle = |\alpha\rangle + i|\beta\rangle$.

$$\begin{aligned}
\langle\gamma|\hat{T}\gamma\rangle &= \langle\gamma\,|\,\hat{T}\,|\,\gamma\rangle \\
&= (\mathsf{a}+i\mathsf{b})^\dagger\mathsf{T}(\mathsf{a}+i\mathsf{b}) \\
&= (\mathsf{a}^\dagger - i\mathsf{b}^\dagger)\mathsf{T}(\mathsf{a}+i\mathsf{b}) \\
&= (\mathsf{a}^\dagger - i\mathsf{b}^\dagger)(\mathsf{Ta}+i\mathsf{Tb}) \\
&= \mathsf{a}^\dagger(\mathsf{Ta}+i\mathsf{Tb}) - i\mathsf{b}^\dagger(\mathsf{Ta}+i\mathsf{Tb}) \\
&= \mathsf{a}^\dagger\mathsf{Ta} + i\mathsf{a}^\dagger\mathsf{Tb} - i\mathsf{b}^\dagger\mathsf{Ta} + \mathsf{b}^\dagger\mathsf{Tb} \\
&= \langle\alpha\,|\,\hat{T}\,|\,\alpha\rangle + i\langle\alpha\,|\,\hat{T}\,|\,\beta\rangle - i\langle\beta\,|\,\hat{T}\,|\,\alpha\rangle + \langle\beta\,|\,\hat{T}\,|\,\beta\rangle \\
&= \langle\alpha|\hat{T}\alpha\rangle + i\langle\alpha|\hat{T}\beta\rangle - i\langle\beta|\hat{T}\alpha\rangle + \langle\beta|\hat{T}\beta\rangle
\end{aligned}$$

Expand the right side now with $|\gamma\rangle = |\alpha\rangle + i|\beta\rangle$.

$$\begin{aligned}
\langle\hat{T}\gamma|\gamma\rangle &= \langle\gamma\,|\,\hat{T}^\dagger\,|\,\gamma\rangle \\
&= (\mathsf{a}+i\mathsf{b})^\dagger\mathsf{T}^\dagger(\mathsf{a}+i\mathsf{b}) \\
&= (\mathsf{a}^\dagger - i\mathsf{b}^\dagger)\mathsf{T}^\dagger(\mathsf{a}+i\mathsf{b}) \\
&= (\mathsf{a}^\dagger\mathsf{T}^\dagger - i\mathsf{b}^\dagger\mathsf{T}^\dagger)(\mathsf{a}+i\mathsf{b}) \\
&= \mathsf{a}^\dagger\mathsf{T}^\dagger(\mathsf{a}+i\mathsf{b}) - i\mathsf{b}^\dagger\mathsf{T}^\dagger(\mathsf{a}+i\mathsf{b}) \\
&= \mathsf{a}^\dagger\mathsf{T}^\dagger\mathsf{a} + i\mathsf{a}^\dagger\mathsf{T}^\dagger\mathsf{b} - i\mathsf{b}^\dagger\mathsf{T}^\dagger\mathsf{a} + \mathsf{b}^\dagger\mathsf{T}^\dagger\mathsf{b} \\
&= \langle\alpha\,|\,\hat{T}^\dagger\,|\,\alpha\rangle + i\langle\alpha\,|\,\hat{T}^\dagger\,|\,\beta\rangle - i\langle\beta\,|\,\hat{T}^\dagger\,|\,\alpha\rangle + \langle\beta\,|\,\hat{T}^\dagger\,|\,\beta\rangle \\
&= \langle\hat{T}\alpha|\alpha\rangle + i\langle\hat{T}\alpha|\beta\rangle - i\langle\hat{T}\beta|\alpha\rangle + \langle\hat{T}\beta|\beta\rangle
\end{aligned}$$

Since $\langle\gamma\,|\,\hat{T}\gamma\rangle = \langle\hat{T}\gamma\,|\,\gamma\rangle$ and $\langle\alpha\,|\,\hat{T}\alpha\rangle = \langle\hat{T}\alpha\,|\,\alpha\rangle$ and $\langle\beta\,|\,\hat{T}\beta\rangle = \langle\hat{T}\beta\,|\,\beta\rangle$,

$$\begin{aligned}
\cancel{\langle\alpha|\hat{T}\alpha\rangle} + i\langle\alpha|\hat{T}\beta\rangle - i\langle\beta|\hat{T}\alpha\rangle + \cancel{\langle\beta|\hat{T}\beta\rangle} &= \cancel{\langle\hat{T}\alpha|\alpha\rangle} + i\langle\hat{T}\alpha|\beta\rangle - i\langle\hat{T}\beta|\alpha\rangle + \cancel{\langle\hat{T}\beta|\beta\rangle} \\
i\langle\alpha|\hat{T}\beta\rangle - i\langle\beta|\hat{T}\alpha\rangle &= i\langle\hat{T}\alpha|\beta\rangle - i\langle\hat{T}\beta|\alpha\rangle \\
\langle\alpha|\hat{T}\beta\rangle - \langle\beta|\hat{T}\alpha\rangle &= \langle\hat{T}\alpha|\beta\rangle - \langle\hat{T}\beta|\alpha\rangle. \qquad (2)
\end{aligned}$$

Add the respective sides of equations (1) and (2).

$$2\langle\alpha|\hat{T}\beta\rangle = 2\langle\hat{T}\alpha|\beta\rangle$$

Therefore, dividing both sides by 2, $\langle\alpha|\hat{T}\beta\rangle = \langle\hat{T}\alpha|\beta\rangle$ for all $|\alpha\rangle$ and $|\beta\rangle$.