Problem A.30

A **unitary transformation** is one for which $\hat{U}^\dagger\hat{U} = 1$.

(a) Show that unitary transformations preserve inner products, in the sense that $\langle\hat{U}\alpha|\hat{U}\beta\rangle = \langle\alpha|\beta\rangle$, for all vectors $|\alpha\rangle$, $|\beta\rangle$.

(b) Show that the eigenvalues of a unitary transformation have modulus 1.

(c) Show that the eigenvectors of a unitary transformation belonging to distinct eigenvalues are orthogonal.

[TYPO: This should be the identity matrix I.]

Solution

Part (a)

Let $\hat{U}$ be a unitary transformation: $\hat{U}^\dagger\hat{U} = \mathsf{I}$. The inner product $\langle\hat{U}\alpha|\hat{U}\beta\rangle$ can be evaluated with respect to an orthonormal basis as shown in Equation A.60 on page 472 of the textbook.

$$\begin{aligned}
\langle\hat{U}\alpha\,|\,\hat{U}\beta\rangle &= (\mathsf{Ua})^\dagger(\mathsf{Ub}) \\
&= (\mathsf{a}^\dagger\mathsf{U}^\dagger)(\mathsf{Ub}) \\
&= \mathsf{a}^\dagger(\mathsf{U}^\dagger\mathsf{U})\mathsf{b} \\
&= \mathsf{a}^\dagger(\mathsf{I})\mathsf{b} \\
&= \mathsf{a}^\dagger\mathsf{b} \\
&= \langle\alpha\,|\,\beta\rangle
\end{aligned}$$

Another way to prove the result is as follows.

$$\begin{aligned}
\langle\hat{U}\alpha\,|\,\hat{U}\beta\rangle &= \langle\hat{U}\alpha\,|\,\hat{U}\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\hat{U}^\dagger\hat{U}\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\mathsf{I}\,|\,\beta\rangle \\
&= \langle\alpha\,|\cdot(\mathsf{I}\,|\,\beta\rangle) \\
&= \langle\alpha\,|\cdot(|\,\beta\rangle) \\
&= \langle\alpha\,|\,\beta\rangle
\end{aligned}$$

Therefore, unitary transformations preserve inner products.

Part (b)

Suppose that λ is an eigenvalue of the unitary transformation $\hat{U}$: $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle$. Then

$$\begin{aligned}
\langle\alpha\,|\,\alpha\rangle &= \langle\alpha\,| \cdot (|\,\alpha\rangle) \\
&= \langle\alpha\,| \cdot (\mathsf{I}\,|\,\alpha\rangle) \\
&= \langle\alpha\,|\,\mathsf{I}\,|\,\alpha\rangle \\
&= \langle\alpha\,|\,\hat{U}^{-1}\hat{U}\,|\,\alpha\rangle \\
&= \langle\alpha\,|\,\hat{U}^{\dagger}\hat{U}\,|\,\alpha\rangle \\
&= \left(\langle\alpha|\hat{U}^{\dagger}\right) \cdot \left(\hat{U}|\alpha\rangle\right) \\
&= \left(\hat{U}|\alpha\rangle\right)^{\dagger} \cdot \left(\hat{U}|\alpha\rangle\right) \\
&= (\lambda|\alpha\rangle)^{\dagger} \cdot (\lambda|\alpha\rangle) \\
&= (\lambda^*\langle\alpha|) \cdot (\lambda|\alpha\rangle) \\
&= \lambda^*\lambda\langle\alpha\,|\,\alpha\rangle.
\end{aligned}$$

Use the fact that $\lambda^*\lambda = |\lambda|^2$.

$$\langle\alpha\,|\,\alpha\rangle = |\lambda|^2\langle\alpha\,|\,\alpha\rangle$$

Since $|\alpha\rangle$ is not the zero vector, $\langle\alpha\,|\,\alpha\rangle \neq 0$. Divide both sides by $\langle\alpha\,|\,\alpha\rangle$.

$$1 = |\lambda|^2$$

Take the square root of both sides.

$$|\lambda| = \pm 1$$

The modulus of a complex number is always nonnegative, so the positive sign is chosen.

$$|\lambda| = 1$$

Therefore, any eigenvalue of a unitary transformation has a modulus of one.

Part (c)

Suppose that λ and μ are distinct eigenvalues of a unitary transformation $\hat{U}$: $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle$ and $\hat{U}|\beta\rangle = \mu|\beta\rangle$. The aim is to show that the eigenvectors, $|\alpha\rangle$ and $|\beta\rangle$, are orthogonal, that is, $\langle\alpha\,|\,\beta\rangle = 0$.

$$\begin{aligned}
\langle\alpha\,|\,\beta\rangle &= \langle\alpha\,|\cdot(|\,\beta\rangle) \\
&= \langle\alpha\,|\cdot(\mathsf{I}\,|\,\beta\rangle) \\
&= \langle\alpha\,|\,\mathsf{I}\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\hat{U}^{-1}\hat{U}\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\hat{U}^{\dagger}\hat{U}\,|\,\beta\rangle \\
&= \left(\langle\alpha|\hat{U}^{\dagger}\right)\cdot\left(\hat{U}|\beta\rangle\right) \\
&= \left(\hat{U}|\alpha\rangle\right)^{\dagger}\cdot\left(\hat{U}|\beta\rangle\right) \\
&= (\lambda|\alpha\rangle)^{\dagger}\cdot(\mu|\beta\rangle) \\
&= (\lambda^{*}\langle\alpha|)\cdot(\mu|\beta\rangle) \\
&= \lambda^{*}\mu\langle\alpha\,|\,\beta\rangle
\end{aligned}$$

Bring both terms to the left side.

$$\langle\alpha\,|\,\beta\rangle - \lambda^{*}\mu\langle\alpha\,|\,\beta\rangle = 0$$

Factor the inner product.

$$(1-\lambda^{*}\mu)\langle\alpha\,|\,\beta\rangle = 0$$

By the zero-product property,

$$1-\lambda^{*}\mu = 0 \quad \text{or} \quad \langle\alpha\,|\,\beta\rangle = 0.$$

The goal now is to show that this equation on the left is false. Multiply both sides of it by λ.

$$\lambda - \lambda^{*}\lambda\mu = 0$$

Use the fact that $\lambda^{*}\lambda = |\lambda|^2 = 1$.

$$\lambda - (1)\mu = 0$$

Solve for λ.

$$\lambda = \mu$$

This contradicts the initial assumption that the eigenvalues are distinct, so $1-\lambda^{*}\mu \neq 0$. Therefore, $\langle\alpha\,|\,\beta\rangle = 0$, which means the eigenvectors corresponding to distinct eigenvalues of a unitary transformation are orthogonal.