Problem A.4

Suppose you start out with a basis $(|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle)$ that is *not* orthonormal. The **Gram–Schmidt procedure** is a systematic ritual for generating from it an orthonormal basis $(|e'_1\rangle, |e'_2\rangle, \ldots, |e'_n\rangle)$. It goes like this:

(i) Normalize the first basis vector (divide by its norm):

$$|e'_1\rangle = \frac{|e_1\rangle}{\|e_1\|}.$$

(ii) Find the projection of the second vector along the first, and subtract it off:

$$|e_2\rangle - \langle e'_1 \,|\, e_2\rangle \, |e'_1\rangle.$$

This vector is orthogonal to $|e'_1\rangle$; normalize it to get $|e'_2\rangle$.

(iii) Subtract from $|e_3\rangle$ its projections along $|e'_1\rangle$ and $|e'_2\rangle$:

$$|e_3\rangle - \langle e'_1 \,|\, e_3\rangle \, |e'_1\rangle - \langle e'_2 \,|\, e_3\rangle \, |e'_2\rangle$$

This is orthogonal to $|e'_1\rangle$ and $|e'_2\rangle$; normalize it to get $|e'_3\rangle$. And so on.

Use the Gram–Schmidt procedure to orthonormalize the 3-space basis $|e_1\rangle = (1+i)\,\hat{\imath} + (1)\,\hat{\jmath} + (i)\,\hat{k}$, $|e_2\rangle = (i)\,\hat{\imath} + (3)\,\hat{\jmath} + (1)\,\hat{k}$, $|e_3\rangle = (0)\,\hat{\imath} + (28)\,\hat{\jmath} + (0)\,\hat{k}$.

Solution

Apply the Gram–Schmidt procedure to obtain an orthonormal basis from the given vectors. Begin by normalizing the first vector.

$$\begin{aligned} |e'_1\rangle = \frac{|e_1\rangle}{\|e_1\|} &= \frac{(1+i)\hat{\mathbf{i}} + \hat{\mathbf{j}} + i\hat{\mathbf{k}}}{\sqrt{(1+i)^*(1+i) + (1)^*(1) + (i)^*(i)}} \\ &= \frac{(1+i)\hat{\mathbf{i}} + \hat{\mathbf{j}} + i\hat{\mathbf{k}}}{\sqrt{(1-i)(1+i) + (1)(1) + (-i)(i)}} \\ &= \frac{(1+i)\hat{\mathbf{i}} + \hat{\mathbf{j}} + i\hat{\mathbf{k}}}{\sqrt{4}} \\ &= \frac{1+i}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \frac{i}{2}\hat{\mathbf{k}} \end{aligned}$$

Calculate the component of the second vector in the direction of the first.

$$\begin{aligned} \langle e'_1 \,|\, e_2\rangle &= \left(\frac{1+i}{2}\right)^* (i) + \left(\frac{1}{2}\right)^* (3) + \left(\frac{i}{2}\right)^* (1) \\ &= \left(\frac{1-i}{2}\right)(i) + \left(\frac{1}{2}\right)(3) + \left(\frac{-i}{2}\right)(1) \\ &= 2 \end{aligned}$$

Subtract this component in the first vector's direction off of the second vector.

$$|e_2\rangle - \langle e_1' \,|\, e_2\rangle \,|e_1'\rangle = (i\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - 2\left(\frac{1+i}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \frac{i}{2}\hat{\mathbf{k}}\right)$$
$$= -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (1-i)\hat{\mathbf{k}}$$

Normalize this new vector.

$$|e_2'\rangle = \frac{|e_2\rangle - \langle e_1' \,|\, e_2\rangle \,|e_1'\rangle}{\|\, |e_2\rangle - \langle e_1' \,|\, e_2\rangle \,|e_1'\rangle \,\|} = \frac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (1-i)\hat{\mathbf{k}}}{\sqrt{(-1)^*(-1) + (2)^*(2) + (1-i)^*(1-i)}}$$
$$= \frac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (1-i)\hat{\mathbf{k}}}{\sqrt{(-1)(-1) + (2)(2) + (1+i)(1-i)}}$$
$$= \frac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (1-i)\hat{\mathbf{k}}}{\sqrt{7}}$$
$$= -\frac{1}{\sqrt{7}}\hat{\mathbf{i}} + \frac{2}{\sqrt{7}}\hat{\mathbf{j}} + \frac{1-i}{\sqrt{7}}\hat{\mathbf{k}}$$

Calculate the component of the third vector in the direction of this new vector.

$$\langle e_2' \,|\, e_3\rangle = \left(-\frac{1}{\sqrt{7}}\right)^*(0) + \left(\frac{2}{\sqrt{7}}\right)^*(28) + \left(\frac{1-i}{\sqrt{7}}\right)^*(0)$$
$$= \left(-\frac{1}{\sqrt{7}}\right)(0) + \left(\frac{2}{\sqrt{7}}\right)(28) + \left(\frac{1+i}{\sqrt{7}}\right)(0)$$
$$= \frac{56}{\sqrt{7}}$$

Calculate the component of the third vector in the direction of the first one.

$$\langle e_1' \,|\, e_3\rangle = \left(\frac{1+i}{2}\right)^*(0) + \left(\frac{1}{2}\right)^*(28) + \left(\frac{i}{2}\right)^*(0)$$
$$= \left(\frac{1-i}{2}\right)(0) + \left(\frac{1}{2}\right)(28) + \left(\frac{-i}{2}\right)(0)$$
$$= 14$$

Subtract these components in the first vector's direction and the new vector's direction off of the third vector.

$$|e_3\rangle - \langle e_1' \,|\, e_3\rangle \,|e_1'\rangle - \langle e_2' \,|\, e_3\rangle \,|e_2'\rangle = \left(0\hat{\mathbf{i}} + 28\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\right) - 14\left(\frac{1+i}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \frac{i}{2}\hat{\mathbf{k}}\right) - \frac{56}{\sqrt{7}}\left(-\frac{1}{\sqrt{7}}\hat{\mathbf{i}} + \frac{2}{\sqrt{7}}\hat{\mathbf{j}} + \frac{1-i}{\sqrt{7}}\hat{\mathbf{k}}\right)$$
$$= (1-7i)\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + (-8+i)\hat{\mathbf{k}}$$

Normalize this second new vector.

$$
\begin{aligned}
|e_3'\rangle = \frac{|e_3\rangle - \langle e_1' \,|\, e_3\rangle\,|e_1'\rangle - \langle e_2' \,|\, e_3\rangle\,|e_2'\rangle}{\|\,|e_3\rangle - \langle e_1' \,|\, e_3\rangle\,|e_1'\rangle - \langle e_2' \,|\, e_3\rangle\,|e_2'\rangle\,\|} &= \frac{(1-7i)\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + (-8+i)\hat{\mathbf{k}}}{\sqrt{(1-7i)^*(1-7i) + (5)^*(5) + (-8+i)^*(-8+i)}} \\
&= \frac{(1-7i)\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + (-8+i)\hat{\mathbf{k}}}{\sqrt{(1+7i)(1-7i) + (5)(5) + (-8-i)(-8+i)}} \\
&= \frac{(1-7i)\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + (-8+i)\hat{\mathbf{k}}}{\sqrt{140}} \\
&= \frac{1-7i}{\sqrt{140}}\hat{\mathbf{i}} + \frac{5}{\sqrt{140}}\hat{\mathbf{j}} + \frac{-8+i}{\sqrt{140}}\hat{\mathbf{k}}
\end{aligned}
$$

Therefore, the three orthonormal basis vectors are

$$
\begin{aligned}
|e_1'\rangle &= \frac{1+i}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \frac{i}{2}\hat{\mathbf{k}} \\
|e_2'\rangle &= -\frac{1}{\sqrt{7}}\hat{\mathbf{i}} + \frac{2}{\sqrt{7}}\hat{\mathbf{j}} + \frac{1-i}{\sqrt{7}}\hat{\mathbf{k}} \\
|e_3'\rangle &= \frac{1-7i}{\sqrt{140}}\hat{\mathbf{i}} + \frac{5}{\sqrt{140}}\hat{\mathbf{j}} + \frac{-8+i}{\sqrt{140}}\hat{\mathbf{k}}.
\end{aligned}
$$