Problem A.6

Find the angle (in the sense of Equation A.28) between the vectors $|\alpha\rangle = (1+i)\,\hat{\imath} + (1)\,\hat{\jmath} + (i)\,\hat{k}$ and $|\beta\rangle = (4-i)\,\hat{\imath} + (0)\,\hat{\jmath} + (2-2i)\,\hat{k}$.

Solution

For typical vectors, **a** and **b**, in three-dimensional space, the dot product is defined as

$$\mathbf{a}\cdot\mathbf{b} = \|a\|\|b\|\cos\theta,$$

where θ is the angle between these vectors. This formula can be generalized to complex vectors, $|\alpha\rangle$ and $|\beta\rangle$, in higher dimensional space using the inner product.

$$\langle\alpha\,|\,\beta\rangle = \|\alpha\|\|\beta\|\cos\theta$$

Square both sides.

$$(\langle\alpha\,|\,\beta\rangle)^2 = \|\alpha\|^2\|\beta\|^2\cos^2\theta$$

Use the fact that for a complex number z, $z^2 = zz^*$.

$$(\langle\alpha\,|\,\beta\rangle)(\langle\alpha\,|\,\beta\rangle)^* = \langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle\cos^2\theta$$

$$\langle\alpha\,|\,\beta\rangle\langle\beta\,|\,\alpha\rangle = \langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle\cos^2\theta$$

Solve for $\cos^2\theta$.

$$\cos^2\theta = \frac{\langle\alpha\,|\,\beta\rangle\langle\beta\,|\,\alpha\rangle}{\langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle}$$

Take the square root of both sides to get Equation A.28 in the textbook.

$$\cos\theta = \sqrt{\frac{\langle\alpha\,|\,\beta\rangle\langle\beta\,|\,\alpha\rangle}{\langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle}} \tag{A.28}$$

The positive root is chosen since $0 \le \theta \le \pi$. The angle will be calculated for the particular case that

$$|\alpha\rangle = (1+i)\hat{\mathbf{i}} + \hat{\mathbf{j}} + i\hat{\mathbf{k}}$$

$$|\beta\rangle = (4-i)\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + (2-2i)\hat{\mathbf{k}}.$$

Plug these vectors into Equation A.28.

$$\begin{aligned}
\cos\theta &= \sqrt{\frac{\langle\alpha\,|\,\beta\rangle\langle\beta\,|\,\alpha\rangle}{\langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle}} \\
&= \sqrt{\frac{[(1+i)^*(4-i)+(1)^*(0)+(i)^*(2-2i)][(4-i)^*(1+i)+(0)^*(1)+(2-2i)^*(i)]}{[(1+i)^*(1+i)+(1)^*(1)+(i)^*(i)][(4-i)^*(4-i)+(0)^*(0)+(2-2i)^*(2-2i)]}} \\
&= \sqrt{\frac{[(1-i)(4-i)+(1)(0)+(-i)(2-2i)][(4+i)(1+i)+(0)(1)+(2+2i)(i)]}{[(1-i)(1+i)+(1)(1)+(-i)(i)][(4+i)(4-i)+(0)(0)+(2+2i)(2-2i)]}} \\
&= \sqrt{\frac{(1-7i)(1+7i)}{(4)(25)}}
\end{aligned}$$

Continue the simplification.

$$\cos\theta = \sqrt{\frac{50}{(4)(25)}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Therefore, the angle between these complex vectors is

$$\theta = \frac{\pi}{4}.$$