Problem A.9

Using the square matrices in Problem A.8, and the column matrices

$$\mathsf{a} = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 2 \\ (1-i) \\ 0 \end{pmatrix},$$

find: (a) Aa, (b) $\mathsf{a}^\dagger\mathsf{b}$, (c) $\tilde{\mathsf{a}}\mathsf{Bb}$, (d) $\mathsf{a}\,\mathsf{b}^\dagger$.

Solution

The square matrices from Problem A.8 are

$$\mathsf{A} = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}.$$

Calculate the desired quantities.

$$\mathsf{Aa} = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} = \begin{pmatrix} (-1)(i) + (1)(2i) + (i)(2) \\ (2)(i) + (0)(2i) + (3)(2) \\ (2i)(i) + (-2i)(2i) + (2)(2) \end{pmatrix} = \begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}$$

$$\begin{aligned} \mathsf{a}^\dagger\mathsf{b} = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix}^{*T} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = \begin{pmatrix} -i \\ -2i \\ 2 \end{pmatrix}^{T} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} &= \begin{pmatrix} -i & -2i & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} \\ &= \big((-i)(2) + (-2i)(1-i) + (2)(0)\big) \\ &= \big(-2-4i\big) \end{aligned}$$

$$\begin{aligned} \tilde{\mathsf{a}}\mathsf{Bb} = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix}^{T} \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} &= \begin{pmatrix} i & 2i & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 2i & 2 \end{pmatrix} \begin{pmatrix} (2)(2) + (0)(1-i) + (-i)(0) \\ (0)(2) + (1)(1-i) + (0)(0) \\ (i)(2) + (3)(1-i) + (2)(0) \end{pmatrix} \\ &= \begin{pmatrix} i & 2i & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1-i \\ 3-i \end{pmatrix} \\ &= \big((i)(4) + (2i)(1-i) + (2)(3-i)\big) \\ &= \big(8+4i\big) \end{aligned}$$

Note that matrix multiplication is associative: $\tilde{\mathsf{a}}\mathsf{Bb} = (\tilde{\mathsf{a}}\mathsf{B})\mathsf{b} = \tilde{\mathsf{a}}(\mathsf{Bb})$.

Calculate the remaining product.

$$\mathsf{a}\,\mathsf{b}^\dagger = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix}^{*T} = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1+i \\ 0 \end{pmatrix}^{T} = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1+i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (i)(2) & i(1+i) & (i)(0) \\ (2i)(2) & (2i)(1+i) & (2i)(0) \\ (2)(2) & (2)(1+i) & (2)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 2i & -1+i & 0 \\ 4i & -2+2i & 0 \\ 4 & 2+2i & 0 \end{pmatrix}$$

$\mathsf{a}\,\mathsf{b}^\dagger$ is the product of a 3×1 matrix and a 1×3 matrix, meaning $\mathsf{a}\,\mathsf{b}^\dagger$ is a 3×3 matrix.