Problem 1.11

[This problem generalizes Example 1.2.] Imagine a particle of mass m and energy E in a potential well $V(x)$, sliding frictionlessly back and forth between the classical turning points (a and b in Figure 1.10).

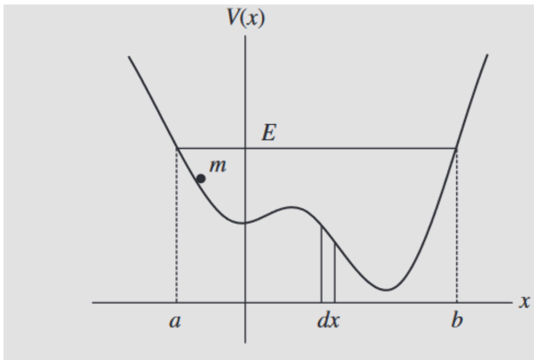


Figure 1.10: Classical particle in a potential well.

Classically, the probability of finding the particle in the range dx (if, for example, you took a snapshot at a random time t) is equal to the fraction of the time T it takes to get from a to b that it spends in the interval dx:

$$\rho(x)\,dx = \frac{dt}{T} = \frac{(dt/dx)\,dx}{T} = \frac{1}{v(x)T}\,dx, \tag{1.41}$$

where $v(x)$ is the speed, and

$$T = \int_0^T dt = \int_a^b \frac{1}{v(x)}\,dx. \tag{1.42}$$

Thus

$$\rho(x) = \frac{1}{v(x)T}. \tag{1.43}$$

This is perhaps the closest analog[22] to $|\Psi|^2$.

(a) Use conservation of energy to express $v(x)$ in terms of E and $V(x)$.

(b) As an example, find $\rho(x)$ for the simple harmonic oscillator, $V(x) = kx^2/2$. Plot $\rho(x)$, and check that it is correctly normalized.

(c) For the classical harmonic oscillator in part (b), find $\langle x\rangle$, $\langle x^2\rangle$, and σ_x.

Solution

By conservation of energy, the sum of potential and kinetic energies must be equal to E, the total mechanical energy.

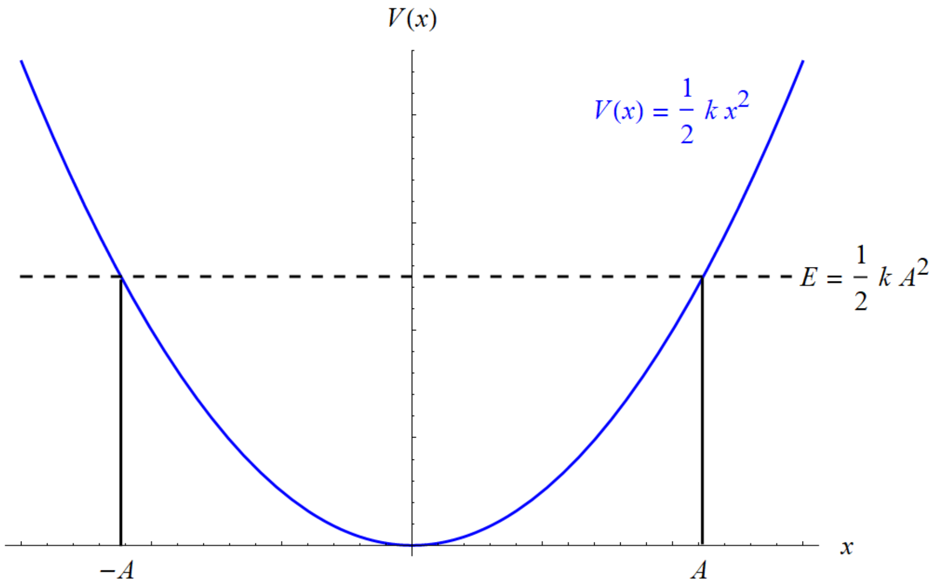
$$E = \text{PE} + \text{KE} = V(x) + \frac{1}{2}mv^2$$

[22] If you like, instead of photos of *one* system at random times, picture an ensemble of such systems, all with the same energy but with random starting positions, and photograph them all at the *same* time. The analysis is identical, but this interpretation is closer to the quantum notion of indeterminacy.

Solve for $v = v(x)$.

$$v(x) = \pm\sqrt{\frac{2}{m}[E - V(x)]}$$

For a simple harmonic oscillator, $V(x) = kx^2/2$, which means the classical turning points are $x = \pm A$, where A is the amplitude of oscillation.



The formula for velocity becomes

$$v(x) = \pm\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)} = \pm\sqrt{\frac{2E - kx^2}{m}},$$

and the probability distribution becomes

$$\rho(x) = \frac{1}{v(x)T} = \frac{1}{v(x)\int_{-A}^{A}\frac{1}{v(x)}\,dx} = \frac{1}{\sqrt{\frac{2E-kx^2}{m}}\int_{-A}^{A}\sqrt{\frac{m}{2E-kx^2}}\,dx} = \frac{1}{\sqrt{\frac{2E-kx^2}{m}}\sqrt{\frac{m}{2E}}\int_{-A}^{A}\frac{dx}{\sqrt{1-\frac{k}{2E}x^2}}}$$

$$= \frac{1}{2\sqrt{1-\frac{k}{2E}x^2}\int_{0}^{A}\frac{dx}{\sqrt{1-\frac{k}{2E}x^2}}}.$$

Make the following trigonometric substitution.

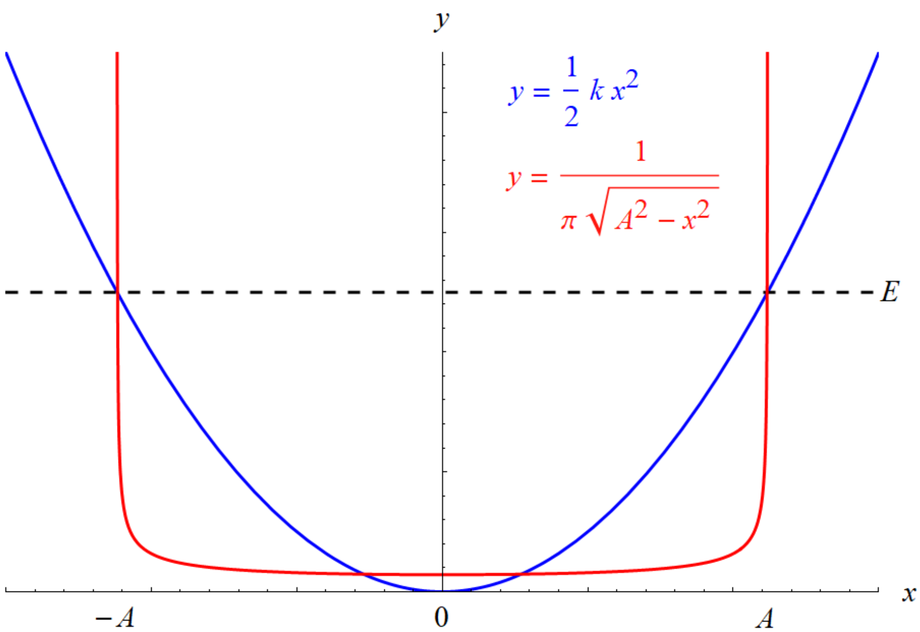
$$x = \sqrt{\frac{2E}{k}}\sin\theta \quad \rightarrow \quad 1 - \frac{k}{2E}x^2 = 1 - \sin^2\theta = \cos^2\theta$$

$$dx = \sqrt{\frac{2E}{k}}\cos\theta\,d\theta$$

As a result,

$$\begin{aligned}
\rho(x) &= \frac{1}{2\sqrt{1-\frac{k}{2E}x^2}\int_0^{\sin^{-1}\left(\sqrt{\frac{k}{2E}}A\right)} \frac{\sqrt{\frac{2E}{k}}\cos\theta\,d\theta}{\cos\theta}} \\
&= \frac{1}{2\sqrt{1-\frac{k}{2E}x^2}\sqrt{\frac{2E}{k}}\int_0^{\sin^{-1}(1)} d\theta} \\
&= \frac{1}{2\sqrt{\frac{2E}{k}-x^2}\left(\frac{\pi}{2}\right)} \\
&= \frac{1}{\pi\sqrt{A^2-x^2}}.
\end{aligned}$$

The domain of this function reveals where it's valid: $|x| < A$. For $|x| \geq A$, $\rho(x) = 0$ because it's impossible for the particle to be here with this energy E.



Observe that the probability amplitude is lowest at the bottom of the well, where the particle is moving fastest, and that the probability amplitude is highest near the top, where the particle comes to a stop. Of course, the probability distribution is normalized because

$$\int_{-A}^{A} \rho(x)\,dx = \int_{-A}^{A} \frac{1}{v(x)T}\,dx = \frac{1}{T}\int_{-A}^{A} \frac{1}{v(x)}\,dx = \frac{1}{\cancel{\int_{-A}^{A}\frac{1}{v(x)}\,dx}}\cancel{\int_{-A}^{A}\frac{1}{v(x)}}\,dx = 1.$$

Check it anyway using the formula found for $\rho(x)$.

$$\int_{-A}^{A} \rho(x)\,dx = \int_{-A}^{A} \frac{1}{\pi\sqrt{A^2-x^2}}\,dx = \frac{1}{\pi}\int_{-A}^{A} \frac{dx}{\sqrt{A^2-x^2}}$$

Make the following trigonometric substitution.

$$x = A\sin\theta \quad \rightarrow \quad A^2 - x^2 = A^2(1-\sin^2\theta) = A^2\cos^2\theta$$
$$dx = A\cos\theta\,d\theta$$

As a result,

$$\begin{aligned}
\int_{-A}^{A} \rho(x)\,dx &= \frac{1}{\pi}\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{A\cos\theta\,d\theta}{A\cos\theta} \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta \\
&= \frac{1}{\pi}(\pi) \\
&= 1.
\end{aligned}$$

Use the probability distribution to calculate the expectation values of x and x^2.

$$\begin{aligned}
\langle x\rangle = \frac{\int_{-A}^{A} x\rho(x)\,dx}{\int_{-A}^{A}\rho(x)\,dx} = \frac{\int_{-A}^{A} x\rho(x)\,dx}{1} &= \int_{-A}^{A} x\rho(x)\,dx \\
&= \int_{-A}^{A} \frac{x}{\pi\sqrt{A^2-x^2}}\,dx \\
&= 0
\end{aligned}$$

This integral is zero because the integrand is an odd function and the interval of integration is symmetric.

$$\begin{aligned}
\langle x^2\rangle = \frac{\int_{-A}^{A} x^2\rho(x)\,dx}{\int_{-A}^{A}\rho(x)\,dx} = \frac{\int_{-A}^{A} x^2\rho(x)\,dx}{1} &= \int_{-A}^{A} x^2\rho(x)\,dx \\
&= \int_{-A}^{A} \frac{x^2}{\pi\sqrt{A^2-x^2}}\,dx \\
&= \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{A^2\sin^2\theta}{\pi(A\cos\theta)}\,(A\cos\theta\,d\theta) \\
&= \frac{A^2}{\pi}\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta \\
&= \frac{A^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta \\
&= \frac{A^2}{2}
\end{aligned}$$

The trigonometric substitution at the top of this page was used. Finally, the standard deviation is

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} \approx 0.707A.$$