Problem 1.12

What if we were interested in the distribution of *momenta* ($p = mv$), for the classical harmonic oscillator (Problem 1.11(b)).

(a) Find the classical probability distribution $\rho(p)$ (note that p ranges from $-\sqrt{2mE}$ to $+\sqrt{2mE}$).

(b) Calculate $\langle p\rangle$, $\langle p^2\rangle$, and σ_p.

(c) What's the *classical* uncertainty product, $\sigma_x\sigma_p$, for this system? Notice that this product can be as small as you like, classically, simply by sending $E \to 0$. But in quantum mechanics, as we shall see in Chapter 2, the energy of a simple harmonic oscillator cannot be less than $\hbar\omega/2$, where $\omega = \sqrt{k/m}$ is the classical frequency. In that case what can you say about the product $\sigma_x\sigma_p$?

Solution

We'll need the formulas of the previous problem.

$$\rho(x)\,dx = \frac{1}{v(x)T}\,dx \tag{1}$$

$$T = \int_a^b \frac{1}{v(x)}dx \tag{2}$$

Assuming there's no friction, conservation of mechanical energy states that the total energy E is the sum of the potential and kinetic energies.

$$E = \text{PE} + \text{KE} = \frac{1}{2}kx^2 + \frac{p^2}{2m}$$

Solve for x in terms of p.

$$x = \pm\sqrt{\frac{2mE - p^2}{mk}}$$

Make this substitution in equations (1) and (2).

$$x = \pm\sqrt{\frac{2mE - p^2}{mk}}$$

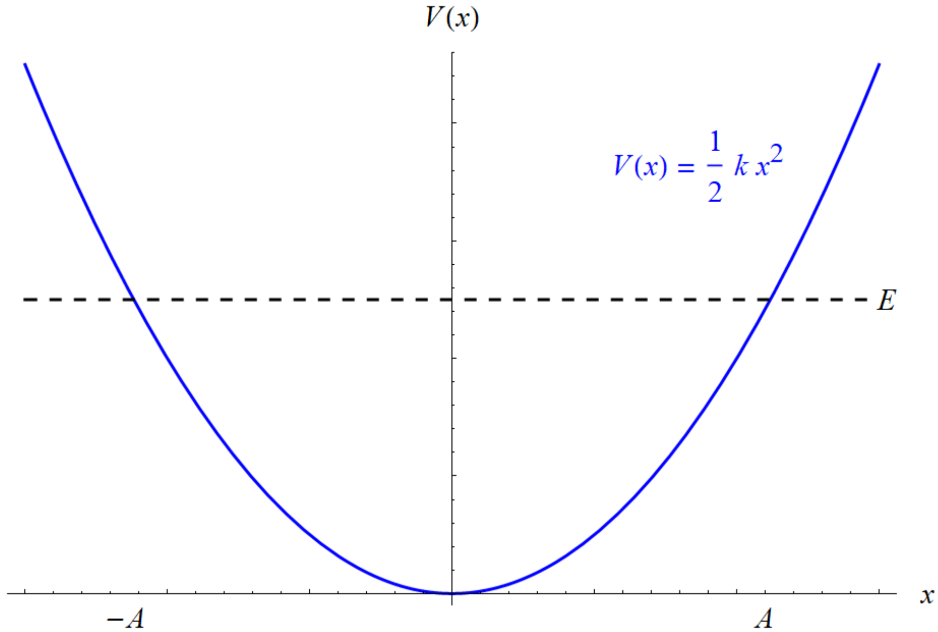
$$dx = \pm\frac{1}{2}\left(\frac{2mE - p^2}{mk}\right)^{-1/2}\left(-\frac{2p}{mk}\right)dp = \mp\frac{p}{\sqrt{mk(2mE - p^2)}}\,dp$$

Consequently, equations (1) and (2) become

$$\rho[x(p)]\left[\mp\frac{p}{\sqrt{mk(2mE - p^2)}}\,dp\right] = \frac{1}{v[x(p)]T}\left[\mp\frac{p}{\sqrt{mk(2mE - p^2)}}\,dp\right] \quad \Rightarrow \quad \rho(p) = \frac{1}{v(p)T}$$

$$T = \int_{p(a)}^{p(b)} \frac{1}{v(p)}\left[\mp\frac{p}{\sqrt{mk(2mE - p^2)}}\,dp\right].$$

The potential function is shown below.



Because there's no friction, the classical turning points are $a = -A$ and $b = A$, where A is the amplitude of oscillation. Also, it takes the same amount of time T for a mass m to go from $x = -A$ to $x = A$ as it does from $x = 0$ to $x = -A$ and back to $x = 0$. The momentum at each of these points is $-\sqrt{2mE}$, 0, and $\sqrt{2mE}$, respectively.

$$\rho(p) = \frac{1}{v(p)\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{1}{v(p)}\left[\mp\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}$$

This quantity in square brackets in front of dp is dx/dp. As the particle is moving to the left, dx is negative and dp is positive, so choose the positive sign for $p = -\sqrt{2mE}$ to $p = 0$. As the particle is moving to the right, dx and dp are positive, so choose the positive sign for $p = 0$ to $p = \sqrt{2mE}$.

$$\rho(p) = \frac{1}{\dfrac{p}{m}\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{m}{p}\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]} = \frac{1}{\dfrac{p}{\sqrt{mk}}\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{dp}{\sqrt{2mE-p^2}}}$$

Make the following trigonometric substitution in the integral.

$$p = \sqrt{2mE}\sin\theta \quad\Rightarrow\quad 2mE - p^2 = 2mE(1-\sin^2\theta) = 2mE\cos^2\theta$$
$$dp = \sqrt{2mE}\cos\theta\,d\theta$$

As a result,

$$\rho(p) = \frac{1}{\dfrac{p}{\sqrt{mk}}\displaystyle\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{\sqrt{2mE}\cos\theta\,d\theta}{\sqrt{2mE}\cos\theta}} = \frac{1}{\dfrac{p}{\sqrt{mk}}\displaystyle\int_{-\pi/2}^{\pi/2} d\theta} = \frac{\sqrt{mk}}{\pi p}.$$

Calculate the expectation value of p.

$$\langle p\rangle = \frac{\displaystyle\int p\rho(x)\,dx}{\displaystyle\int \rho(x)\,dx} = \frac{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} p\rho[x(p)]\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \rho[x(p)]\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]} = \frac{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} p\left(\frac{\sqrt{mk}}{\pi p}\right)\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{\sqrt{mk}}{\pi p}\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}$$

$$= \frac{\displaystyle\frac{1}{\pi}\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{p}{\sqrt{2mE-p^2}}\,dp}{\displaystyle\frac{1}{\pi}\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{dp}{\sqrt{2mE-p^2}}}$$

$$= \frac{\frac{1}{\pi}(0)}{\frac{1}{\pi}(\pi)}$$

$$= 0$$

The integral on top is zero because the integrand is odd and the integration interval is symmetric. Now calculate the expectation value of p^2.

$$\langle p^2\rangle = \frac{\displaystyle\int p^2\rho(x)\,dx}{\displaystyle\int \rho(x)\,dx} = \frac{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} p^2\rho[x(p)]\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \rho[x(p)]\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]} = \frac{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} p^2\left(\frac{\sqrt{mk}}{\pi p}\right)\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}{\displaystyle\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{\sqrt{mk}}{\pi p}\left[\frac{p}{\sqrt{mk(2mE-p^2)}}\,dp\right]}$$

$$= \frac{\displaystyle\frac{1}{\pi}\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{p^2}{\sqrt{2mE-p^2}}\,dp}{\displaystyle\frac{1}{\pi}\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{dp}{\sqrt{2mE-p^2}}}$$

$$= \frac{\displaystyle\frac{1}{\pi}\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{2mE\sin^2\theta}{\sqrt{2mE}\cos\theta}\,(\sqrt{2mE}\cos\theta\,d\theta)}{\displaystyle\frac{1}{\pi}\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{\sqrt{2mE}\cos\theta\,d\theta}{\sqrt{2mE}\cos\theta}}$$

$$= \frac{\displaystyle\frac{2mE}{\pi}\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta}{\displaystyle\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta}$$

$$= \frac{\frac{2mE}{\pi}\left(\frac{\pi}{2}\right)}{\frac{1}{\pi}(\pi)}$$

$$= mE$$

Finally, calculate the standard deviation.

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{mE}$$

$\sigma_x = A/\sqrt{2}$ was determined in the previous problem. The uncertainty product is

$$\sigma_x \sigma_p = \left(\frac{A}{\sqrt{2}}\right)(\sqrt{mE}) = A\sqrt{\frac{mE}{2}}.$$

Notice that it goes to zero as $E \to 0$. For the quantum mechanical simple harmonic oscillator, though,

$$E \geq \frac{\hbar\omega}{2},$$

which means

$$\sigma_x \sigma_p \geq A\sqrt{\frac{m\left(\frac{\hbar\omega}{2}\right)}{2}} = \frac{A\sqrt{m\hbar\omega}}{4}.$$