Problem 1.15

Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 \, dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation (with the same $V(x)$), Ψ_1 and Ψ_2.

Solution

The governing equation for the wave function $\Psi(x,t)$ is Schrödinger's equation.

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t)$$

Take the complex conjugate of both sides to get the corresponding equation for Ψ^*.

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi^*(x,t)$$

Suppose that Ψ_1 and Ψ_2 are two solutions to the Schrödinger equation.

$$\frac{\partial \Psi_1}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi_1}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi_1(x,t) \qquad \frac{\partial \Psi_2}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi_2}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi_2(x,t)$$

$$\frac{\partial \Psi_1^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi_1^*(x,t) \qquad \frac{\partial \Psi_2^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi_2^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi_2^*(x,t)$$

Now consider the derivative of the integral in question.

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^*\Psi_2\, dx &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\Psi_1^*\Psi_2)\, dx \\
&= \int_{-\infty}^{\infty} \left(\frac{\partial \Psi_1^*}{\partial t}\Psi_2 + \Psi_1^*\frac{\partial \Psi_2}{\partial t}\right) dx \\
&= \int_{-\infty}^{\infty} \left[\left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\Psi_1^*\right)\Psi_2 + \Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi_2}{\partial x^2} - \frac{i}{\hbar}V\Psi_2\right)\right] dx \\
&= \int_{-\infty}^{\infty} \left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi_1^*}{\partial x^2}\Psi_2 + \cancel{\frac{i}{\hbar}V\Psi_1^*\Psi_2} + \frac{i\hbar}{2m}\Psi_1^*\frac{\partial^2 \Psi_2}{\partial x^2} - \cancel{\frac{i}{\hbar}V\Psi_1^*\Psi_2}\right) dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left(\Psi_1^*\frac{\partial^2 \Psi_2}{\partial x^2} - \frac{\partial^2 \Psi_1^*}{\partial x^2}\Psi_2\right) dx \\
&= \frac{i\hbar}{2m}\left(\int_{-\infty}^{\infty} \Psi_1^*\frac{\partial^2 \Psi_2}{\partial x^2}\, dx - \int_{-\infty}^{\infty} \frac{\partial^2 \Psi_1^*}{\partial x^2}\Psi_2\, dx\right) \\
&= \frac{i\hbar}{2m}\left[\left(\underbrace{\Psi_1^*\frac{\partial \Psi_2}{\partial x}\bigg|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}\, dx\right) - \left(\underbrace{\frac{\partial \Psi_1^*}{\partial x}\Psi_2\bigg|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}\, dx\right)\right] \\
&= \frac{i\hbar}{2m}\left(-\cancel{\int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}}\, dx + \cancel{\int_{-\infty}^{\infty} \frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}}\, dx\right) \\
&= 0
\end{aligned}$$