Problem 1.5

Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.)

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Solution

Normalize the wave function by requiring the integral of $|\Psi(x,t)|^2$ over all x to be 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} \Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} (Ae^{-\lambda|x|}e^{-i\omega t})(Ae^{-\lambda|x|}e^{i\omega t})\,dx \\
&= A^2\int_{-\infty}^{\infty} e^{-2\lambda|x|}\,dx \\
&= A^2\left[\int_{-\infty}^{0} e^{-2\lambda(-x)}\,dx + \int_{0}^{\infty} e^{-2\lambda(x)}\,dx\right] \\
&= A^2\left(\int_{-\infty}^{0} e^{2\lambda x}\,dx + \int_{0}^{\infty} e^{-2\lambda x}\,dx\right) \\
&= A^2\left(\frac{1}{2\lambda}e^{2\lambda x}\Big|_{-\infty}^{0} + \frac{1}{-2\lambda}e^{-2\lambda x}\Big|_{0}^{\infty}\right) \\
&= A^2\left[\frac{1}{2\lambda}(1-0) + \frac{1}{-2\lambda}(0-1)\right] \\
&= A^2\left(\frac{1}{\lambda}\right)
\end{aligned}$$

As a result,

$$A = \sqrt{\lambda}.$$

With this value of A, the wave function becomes

$$\Psi(x,t) = \sqrt{\lambda}e^{-\lambda|x|}e^{-i\omega t}.$$

According to Born's interpretation, the probability distribution for the particle's position at time t is

$$|\Psi(x,t)|^2 = \Psi(x,t)\Psi^*(x,t) = \left(\sqrt{\lambda}e^{-\lambda|x|}e^{-i\omega t}\right)\left(\sqrt{\lambda}e^{-\lambda|x|}e^{i\omega t}\right) = \lambda e^{-2\lambda|x|}.$$

Use it to calculate the expectation values of x and x^2.

$$\begin{aligned}
\langle x \rangle = \frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x\Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x(\sqrt{\lambda}e^{-\lambda|x|}e^{-i\omega t})(\sqrt{\lambda}e^{-\lambda|x|}e^{i\omega t})\,dx \\
&= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}\,dx \\
&= 0
\end{aligned}$$

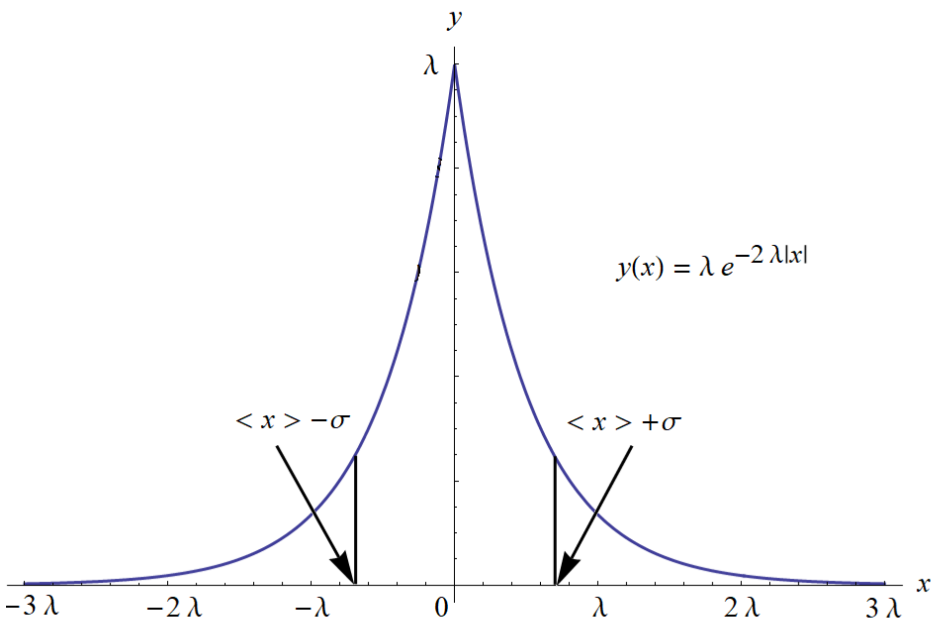
This last integral is zero because the integral is of an odd function over a symmetric interval.

$$\begin{aligned}
\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx}{\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x^2\Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x^2(\sqrt{\lambda}e^{-\lambda|x|}e^{-i\omega t})(\sqrt{\lambda}e^{-\lambda|x|}e^{i\omega t})\,dx \\
&= \lambda \int_{-\infty}^{\infty} x^2e^{-2\lambda|x|}\,dx \\
&= 2\lambda \int_{0}^{\infty} x^2e^{-2\lambda|x|}\,dx \\
&= 2\lambda \int_{0}^{\infty} x^2e^{-2\lambda x}\,dx \\
&= 2\lambda \int_{0}^{\infty} \frac{1}{4}\frac{\partial^2}{\partial\lambda^2}\left(e^{-2\lambda x}\right)dx \\
&= \frac{\lambda}{2}\frac{d^2}{d\lambda^2}\int_{0}^{\infty} e^{-2\lambda x}\,dx \\
&= \frac{\lambda}{2}\frac{d^2}{d\lambda^2}\left(\frac{1}{-2\lambda}e^{-2\lambda x}\bigg|_0^\infty\right) \\
&= \frac{\lambda}{2}\frac{d^2}{d\lambda^2}\left[\frac{1}{-2\lambda}(0-1)\right] \\
&= \frac{\lambda}{4}\frac{d^2}{d\lambda^2}\left(\frac{1}{\lambda}\right) \\
&= \frac{\lambda}{4}\left(\frac{2}{\lambda^3}\right) \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

The standard deviation is then

$$\begin{aligned}\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2}\\ &= \sqrt{\frac{1}{2\lambda^2}}\\ &= \frac{1}{\sqrt{2}\lambda}.\end{aligned}$$

Below is a plot of the probability distribution versus x.



The probability of finding the particle outside $(\langle x\rangle - \sigma, \langle x\rangle + \sigma)$ is

$$\begin{aligned}P &= \int_{-\infty}^{\langle x\rangle-\sigma} |\Psi(x,t)|^2\,dx + \int_{\langle x\rangle+\sigma}^{\infty} |\Psi(x,t)|^2\,dx\\ &= \int_{-\infty}^{-1/(\sqrt{2}\lambda)} \lambda e^{-2\lambda|x|}\,dx + \int_{1/(\sqrt{2}\lambda)}^{\infty} \lambda e^{-2\lambda|x|}\,dx\\ &= \lambda\int_{-\infty}^{-1/(\sqrt{2}\lambda)} e^{-2\lambda(-x)}\,dx + \lambda\int_{1/(\sqrt{2}\lambda)}^{\infty} e^{-2\lambda(x)}\,dx\\ &= \lambda\left[\int_{-\infty}^{-1/(\sqrt{2}\lambda)} e^{2\lambda x}\,dx + \int_{1/(\sqrt{2}\lambda)}^{\infty} e^{-2\lambda x}\,dx\right]\\ &= \lambda\left[\frac{1}{2\lambda}e^{2\lambda x}\bigg|_{-\infty}^{-1/(\sqrt{2}\lambda)} + \frac{1}{-2\lambda}e^{-2\lambda x}\bigg|_{1/(\sqrt{2}\lambda)}^{\infty}\right]\\ &= \lambda\left[\frac{1}{2\lambda}(e^{-\sqrt{2}} - 0) + \frac{1}{-2\lambda}(0 - e^{-\sqrt{2}})\right]\\ &= \lambda\left(\frac{1}{\lambda}e^{-\sqrt{2}}\right)\\ &= e^{-\sqrt{2}} \approx 0.243\end{aligned}$$