Problem 1.6

Why can't you do integration-by-parts directly on the middle expression in Equation 1.29—pull the time derivative over onto x, note that $\partial x/\partial t = 0$, and conclude that $d\langle x\rangle/dt = 0$?

Solution

Equation 1.29 is at the bottom of page 16.

$$\frac{d\langle x\rangle}{dt} = \int_{-\infty}^{\infty} x\frac{\partial}{\partial t}|\Psi(x,t)|^2\,dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \tag{1.29}$$

The integral in the middle can't be done by parts because the derivative and integral are taken with respect to different variables, t and x. The Schrödinger equation and its complex-conjugated equation are used to eliminate the time derivatives in favor of x so that integration-by-parts can be applied.

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[x|\Psi(x,t)|^2]\,dx \\
&= \int_{-\infty}^{\infty} x\frac{\partial}{\partial t}[\Psi^*(x,t)\Psi(x,t)]\,dx \\
&= \int_{-\infty}^{\infty} x\left(\frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty} x\left[\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\Psi + \Psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\right]dx \\
&= \int_{-\infty}^{\infty} x\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{i}{\hbar}V\Psi^*\Psi} + \frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V\Psi^*\Psi}\right)dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx
\end{aligned}$$

At this point the integral can be done by parts.