Problem 1.9

A particle of mass m has the wave function

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where A and a are positive real constants.

(a) Find A.

(b) For what potential energy function, $V(x)$, is this a solution to the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

Solution

Part (a)

Normalize the wave equation by requiring the integral of $|\Psi(x,t)|^2$ over all x to be 1. Use the gaussian integral inside the back cover to evaluate the integral.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} \Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} \{Ae^{-a[(mx^2/\hbar)+it]}\}\{Ae^{-a[(mx^2/\hbar)-it]}\}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-x^2/\left[\sqrt{\hbar/(2am)}\right]^2}\,dx \\
&= 2A^2 \int_{0}^{\infty} e^{-x^2/\left[\sqrt{\hbar/(2am)}\right]^2}\,dx \\
&= 2A^2 \cdot \sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{2am}}}{2}\right) \\
&= A^2\sqrt{\frac{\pi\hbar}{2am}}
\end{aligned}$$

Solve for A.

$$A = \sqrt[4]{\frac{2am}{\pi\hbar}}$$

Therefore, the normalized wave function is

$$\Psi(x,t) = \sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}.$$

Part (b)

The governing equation for the wave function is the Schrödinger equation.

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t) \tag{1}$$

Solve for $V(x,t)$, substitute the wave function, and then simplify.

$$\begin{aligned}
V(x,t) &= \frac{\hbar}{i}\frac{1}{\Psi(x,t)}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial \Psi}{\partial t}\right) \\
&= \frac{\hbar}{i}\sqrt[4]{\frac{\pi\hbar}{2am}}e^{a[(mx^2/\hbar)+it]}\left[\frac{i\hbar}{2m}\cdot\sqrt[4]{\frac{2am}{\pi\hbar}}\frac{\partial^2}{\partial x^2}\left(e^{-a[(mx^2/\hbar)+it]}\right) - \sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}(-ia)\right] \\
&= \hbar e^{a[(mx^2/\hbar)+it]}\left[\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-a[(mx^2/\hbar)+it]}\right) + ae^{-a[(mx^2/\hbar)+it]}\right] \\
&= \hbar e^{a[(mx^2/\hbar)+it]}\left[\frac{\hbar}{2m}\frac{\partial}{\partial x}\left(-\frac{2am}{\hbar}xe^{-a[(mx^2/\hbar)+it]}\right) + ae^{-a[(mx^2/\hbar)+it]}\right] \\
&= \hbar e^{a[(mx^2/\hbar)+it]}\left\{\frac{\hbar}{2m}\left[-\frac{2am}{\hbar}\left(1 - \frac{2am}{\hbar}x^2\right)e^{-a[(mx^2/\hbar)+it]}\right] + ae^{-a[(mx^2/\hbar)+it]}\right\} \\
&= \hbar\left\{a\left(\frac{2am}{\hbar}x^2 - 1\right) + a\right\} \\
&= 2ma^2x^2
\end{aligned}$$

Part (c)

According to Born's interpretation, $|\Psi(x,t)|^2$ represents the probability distribution for the particle's position at time t. Use it to calculate the expectation value of x.

$$\begin{aligned}
\langle x\rangle = \frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x\Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x\left[\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}\right]\left[\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)-it]}\right]dx \\
&= \sqrt{\frac{2am}{\pi\hbar}}\int_{-\infty}^{\infty} xe^{-2amx^2/\hbar}\,dx \\
&= 0
\end{aligned}$$

This last result comes from the fact that the integral of an odd function over a symmetric interval is zero.

Use the probability distribution to calculate the expectation value of x^2. Use the gaussian integral inside the back cover to evaluate the integral.

$$\begin{aligned}
\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2\,dx}{\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x^2 \Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x^2 \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)+it]}\right]\left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right] dx \\
&= \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar}\,dx \\
&= 2\sqrt{\frac{2am}{\pi\hbar}} \int_{0}^{\infty} x^2 e^{-2amx^2/\hbar}\,dx \\
&= 2\sqrt{\frac{2am}{\pi\hbar}} \int_{0}^{\infty} x^2 e^{-x^2/\left[\sqrt{\hbar/(2am)}\right]^2}\,dx \\
&= 2\sqrt{\frac{2am}{\pi\hbar}} \cdot \sqrt{\pi}\,\frac{(2)!}{1!}\left(\frac{\sqrt{\frac{\hbar}{2am}}}{2}\right)^3 \\
&= \frac{\hbar}{4am}
\end{aligned}$$

According to Ehrenfest's theorem,

$$\langle p \rangle = m\langle v \rangle = m\frac{d\langle x \rangle}{dt} = 0.$$

Check this result by calculating $\langle p \rangle$ manually.

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial \Psi}{\partial x}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial \Psi}{\partial x}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right]\frac{\partial}{\partial x}\left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)+it]}\right] dx \\
&= -i\hbar\int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right]\left[\sqrt[4]{\frac{2am}{\pi\hbar}}\left(-\frac{2am}{\hbar}x\right) e^{-a[(mx^2/\hbar)+it]}\right] dx \\
&= i\sqrt{\frac{2am}{\pi\hbar}}(2am)\underbrace{\int_{-\infty}^{\infty} x e^{-2amx^2/\hbar}\,dx}_{=0} \\
&= 0
\end{aligned}$$

Now calculate $\langle p^2 \rangle$ with the provided gaussian integrals inside the back cover of the book.

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(i^2\hbar^2\frac{\partial^2}{\partial x^2}\right) \Psi(x,t)\,dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial^2 \Psi}{\partial x^2}\,dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right] \frac{\partial^2}{\partial x^2}\left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)+it]}\right] dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right] \frac{\partial}{\partial x}\left[\sqrt[4]{\frac{2am}{\pi\hbar}} \left(-\frac{2am}{\hbar}x\right) e^{-a[(mx^2/\hbar)+it]}\right] dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[(mx^2/\hbar)-it]}\right] \left[\sqrt[4]{\frac{2am}{\pi\hbar}} \left(-\frac{2am}{\hbar}\right)\left(1-\frac{2am}{\hbar}x^2\right) e^{-a[(mx^2/\hbar)+it]}\right] dx \\
&= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}}\left(\frac{2am}{\hbar}\right)\int_{-\infty}^{\infty}\left(1-\frac{2am}{\hbar}x^2\right)e^{-2amx^2/\hbar}\,dx \\
&= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}}\left(\frac{2am}{\hbar}\right)\left(\int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\,dx - \frac{2am}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar}\,dx\right) \\
&= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}}\left(\frac{2am}{\hbar}\right)\left(2\int_{0}^{\infty} e^{-2amx^2/\hbar}\,dx - \frac{4am}{\hbar}\int_{0}^{\infty} x^2 e^{-2amx^2/\hbar}\,dx\right) \\
&= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}}\left(\frac{4am}{\hbar}\right)\left\{\int_{0}^{\infty} e^{-x^2/\left[\sqrt{\hbar/(2am)}\right]^2}\,dx - \frac{2am}{\hbar}\int_{0}^{\infty} x^2 e^{-x^2/\left[\sqrt{\hbar/(2am)}\right]^2}\,dx\right\} \\
&= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}}\left(\frac{4am}{\hbar}\right)\left\{\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{2am}}}{2}\right) - \frac{2am}{\hbar}\cdot\sqrt{\pi}\,\frac{(2)!}{1!}\left(\frac{\sqrt{\frac{\hbar}{2am}}}{2}\right)^3\right\} \\
&= \hbar^2\left(\frac{4am}{\hbar}\right)\left(\frac{1}{2}-\frac{1}{4}\right) \\
&= \hbar am
\end{aligned}$$

Part (d)

Use the results in part (c) to determine the standard deviations in x and p.

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{4am} - 0^2} = \frac{1}{2}\sqrt{\frac{\hbar}{am}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar am - 0^2} = \sqrt{\hbar am}$$

The product of σ_x and σ_p is

$$\sigma_x\sigma_p = \frac{1}{2}\sqrt{\frac{\hbar}{am}}\sqrt{\hbar am} = \frac{\hbar}{2},$$

which is consistent with Heisenberg's uncertainty principle ($\sigma_x\sigma_p \geq \hbar/2$).