Problem 2.14

In the ground state of the harmonic oscillator, what is the probability (correct to three significant digits) of finding the particle outside the classically allowed region? *Hint:* Classically, the energy of an oscillator is $E = (1/2)ka^2 = (1/2)m\omega^2 a^2$, where a is the amplitude. So the "classically allowed region" for an oscillator of energy E extends from $-\sqrt{2E/m\omega^2}$ to $+\sqrt{2E/m\omega^2}$. Look in a math table under "Normal Distribution" or "Error Function" for the numerical value of the integral, or evaluate it by computer.

Solution

According to Born's interpretation, the probability distribution for the particle's position in the ground state at time t is given by $|\Psi_0(x,t)|^2$. The probability that the particle lies in the classically allowed region is

$$\int_{-a}^{a} |\Psi_0(x,t)|^2 \, dx,$$

so the probability that it's not in this region is

$$\begin{aligned}
1 - \int_{-a}^{a} |\Psi_0(x,t)|^2 \, dx &= 1 - \int_{-a}^{a} \Psi_0(x,t)\Psi_0^*(x,t) \, dx \\
&= 1 - \int_{-a}^{a} [\psi_0(x)e^{-iE_0t/\hbar}][\psi_0(x)e^{iE_0t/\hbar}] \, dx \\
&= 1 - \int_{-a}^{a} [\psi_0(x)]^2 \, dx \\
&= 1 - \int_{-a}^{a} \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]^2 dx \\
&= 1 - \int_{-a}^{a} \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega}{\hbar}x^2\right)\right] dx \\
&= 1 - \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-a}^{a} \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx.
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
d\xi &= \sqrt{\frac{m\omega}{\hbar}}\,dx \quad \rightarrow \quad dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi
\end{aligned}$$

Consequently,

$$\begin{aligned}
1 - \int_{-a}^{a} |\Psi_0(x,t)|^2 \, dx &= 1 - \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\sqrt{\frac{m\omega}{\hbar}}a}^{\sqrt{\frac{m\omega}{\hbar}}a} e^{-\xi^2} \left(\sqrt{\frac{\hbar}{m\omega}}\,d\xi\right) \\
&= 1 - \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{m\omega}{\hbar}}a}^{\sqrt{\frac{m\omega}{\hbar}}a} e^{-\xi^2} \, d\xi \\
&= 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{\frac{m\omega}{\hbar}}a} e^{-\xi^2} \, d\xi.
\end{aligned}$$

There's a known special function called the error funcion, which is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\,dx,$$

so the probability can be written as

$$1 - \int_{-a}^{a} |\Psi_0(x,t)|^2\,dx = 1 - \operatorname{erf}\left(\sqrt{\frac{m\omega}{\hbar}}a\right).$$

Since the energy of the ground state is known, this argument can be simplified.

$$\frac{\hbar\omega}{2} = \frac{1}{2} m\omega^2 a^2$$

Solve for a.

$$a = \sqrt{\frac{\hbar}{m\omega}}$$

Therefore, the probability that the particle lies outside the classically allowed region in the ground state is

$$1 - \int_{-a}^{a} |\Psi_0(x,t)|^2\,dx = 1 - \operatorname{erf} 1 \approx 0.157.$$