Problem 2.15

Use the recursion formula (Equation 2.85) to work out $H_5(\xi)$ and $H_6(\xi)$. Invoke the convention that the coefficient of the highest power of ξ is 2^n to fix the overall constant.

Solution

The recursion formula in Equation 2.85 is

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j, \tag{2.85}$$

and it gives the coefficients to the power series solution,

$$h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j,$$

of Hermite's ODE. Note that a_0 and a_1 are arbitrary.

<u>$\boldsymbol{n = 0}$</u>

Suppose that $n = 0$.

$$a_{j+2} = \frac{-2(-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 0$, which means a_2, a_4, ... are zero. Set $a_1 = 0$ in order to terminate the odd powers in the series. What remains then is a_0. Require that

$$a_0 = 2^0 = 1$$

to conform with the 2^n rule.

$$h(\xi) = 1$$

<u>$\boldsymbol{n = 1}$</u>

Suppose that $n = 1$.

$$a_{j+2} = \frac{-2(1-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 1$, which means a_3, a_5, ... are zero. Set $a_0 = 0$ in order to terminate the even powers in the series. What remains then is a_1. Require that

$$a_1 = 2^1 = 2$$

to conform with the 2^n rule.

$$h(\xi) = 2\xi$$

$n = 2$

Suppose that $n = 2$.

$$a_{j+2} = \frac{-2(2-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 2$, which means a_4, a_6, ... are zero. Set $a_1 = 0$ in order to terminate the odd powers in the series. What remains then are a_0 and a_2. Require that

$$a_2 = \frac{-2(2)}{(1)(2)} a_0 = 2^2 \quad \Rightarrow \quad \begin{cases} a_0 = -2 \\ a_2 = 4 \end{cases}$$

to conform with the 2^n rule.

$$h(\xi) = 4\xi^2 - 2$$

$n = 3$

Suppose that $n = 3$.

$$a_{j+2} = \frac{-2(3-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 3$, which means a_5, a_7, ... are all zero. Set $a_0 = 0$ in order to terminate the even powers in the series. What remains then are a_1 and a_3. Require that

$$a_3 = \frac{-2(2)}{(2)(3)} a_1 = 2^3 \quad \Rightarrow \quad \begin{cases} a_1 = -12 \\ a_3 = 8 \end{cases}$$

to conform with the 2^n rule.

$$h(\xi) = 8\xi^3 - 12\xi$$

$n = 4$

Suppose that $n = 4$.

$$a_{j+2} = \frac{-2(4-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 4$, which means a_6, a_8, ... are all zero. Set $a_1 = 0$ in order to terminate the odd powers in the series. What remains then are a_0, a_2, and a_4. Require that

$$a_4 = \frac{-2(2)}{(3)(4)} a_2 = \frac{-2(2)}{(3)(4)} \left[\frac{-2(4)}{(1)(2)} a_0 \right] = 2^4 \quad \Rightarrow \quad \begin{cases} a_0 = 12 \\ a_2 = -48 \\ a_4 = 16 \end{cases}$$

to conform with the 2^n rule.

$$h(\xi) = 16\xi^4 - 48\xi^2 + 12$$

$n = 5$

Suppose that $n = 5$.

$$a_{j+2} = \frac{-2(5-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 5$, which means a_7, a_9, ... are all zero. Set $a_0 = 0$ in order to terminate the even powers in the series. What remains then are a_1, a_3, and a_5. Require that

$$a_5 = \frac{-2(2)}{(4)(5)} a_3 = \frac{-2(2)}{(4)(5)} \left[\frac{-2(4)}{(2)(3)} a_1\right] = 2^5 \quad \Rightarrow \quad \begin{cases} a_1 = 60 \\ a_3 = -160 \\ a_5 = 32 \end{cases}$$

to conform with the 2^n rule.

$$h(\xi) = 32\xi^5 - 160\xi^3 + 60\xi$$

$n = 6$

Suppose that $n = 6$.

$$a_{j+2} = \frac{-2(6-j)}{(j+1)(j+2)} a_j$$

The numerator is zero if $j = 6$, which means a_8, a_{10}, ... are all zero. Set $a_1 = 0$ in order to terminate the odd powers in the series. What remains then are a_0, a_2, a_4, and a_6. Require that

$$a_6 = \frac{-2(2)}{(5)(6)} a_4 = \frac{-2(2)}{(5)(6)} \left[\frac{-2(4)}{(3)(4)} a_2\right] = \frac{-2(2)}{(5)(6)} \left\{\frac{-2(4)}{(3)(4)} \left[\frac{-2(6)}{(1)(2)} a_0\right]\right\} = 2^6 \quad \Rightarrow \quad \begin{cases} a_0 = -120 \\ a_2 = 720 \\ a_4 = -480 \\ a_6 = 64 \end{cases}$$

to conform with the 2^n rule.

$$h(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$$