Problem 2.16

In this problem we explore some of the more useful theorems (stated without proof) involving Hermite polynomials.

(a) The **Rodrigues** formula says that

$$H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}. \tag{2.87}$$

Use it to derive H_3 and H_4.

(b) The following recursion relation gives you H_{n+1} in terms of the two preceding Hermite polynomials:

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi). \tag{2.88}$$

Use it, together with your answer in (a), to obtain H_5 and H_6.

(c) If you differentiate an nth-order polynomial, you get a polynomial of order $(n-1)$. For the Hermite polynomials, in fact,

$$\frac{dH_n}{d\xi} = 2nH_{n-1}(\xi). \tag{2.89}$$

Check this, by differentiating H_5 and H_6.

(d) $H_n(\xi)$ is the nth z-derivative, at $z = 0$, of the **generating function** $\exp(-z^2 + 2z\xi)$; or, to put it another way, it is the coefficient of $z^n/n!$ in the Taylor series expansion for this function:

$$e^{-z^2+2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi). \tag{2.90}$$

Use this to obtain H_1, H_2, and H_3.

Solution

Part (a)

Use the Rodrigues formula with $n = 3$ to calculate H_3.

$$\begin{aligned}
H_3(\xi) &= (-1)^3 e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} \\
&= -e^{\xi^2}\frac{d^3}{d\xi^3}(e^{-\xi^2}) \\
&= -e^{\xi^2}\frac{d^2}{d\xi^2}(-2\xi e^{-\xi^2}) \\
&= -e^{\xi^2}\frac{d}{d\xi}(-2e^{-\xi^2} + 4\xi^2 e^{-\xi^2}) \\
&= -e^{\xi^2}(4\xi e^{-\xi^2} + 8\xi e^{-\xi^2} - 8\xi^3 e^{-\xi^2}) \\
&= 8\xi^3 - 12\xi
\end{aligned}$$

Use the Rodrigues formula with $n = 4$ to calculate H_4.

$$\begin{aligned}
H_4(\xi) &= (-1)^4 e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} \\
&= e^{\xi^2}\frac{d^4}{d\xi^4}(e^{-\xi^2}) \\
&= e^{\xi^2}\frac{d^3}{d\xi^3}(-2\xi e^{-\xi^2}) \\
&= e^{\xi^2}\frac{d^2}{d\xi^2}(-2e^{-\xi^2} + 4\xi^2 e^{-\xi^2}) \\
&= e^{\xi^2}\frac{d}{d\xi}(4\xi e^{-\xi^2} + 8\xi e^{-\xi^2} - 8\xi^3 e^{-\xi^2}) \\
&= e^{\xi^2}\frac{d}{d\xi}(12\xi e^{-\xi^2} - 8\xi^3 e^{-\xi^2}) \\
&= e^{\xi^2}(12e^{-\xi^2} - 24\xi^2 e^{-\xi^2} - 24\xi^2 e^{-\xi^2} + 16\xi^4 e^{-\xi^2}) \\
&= 16\xi^4 - 48\xi^2 + 12
\end{aligned}$$

Part (b)

Use the recursion relation with $n = 4$ to determine H_5.

$$\begin{aligned}
H_5(\xi) &= 2\xi H_4(\xi) - 2(4)H_3(\xi) \\
&= 2\xi(16\xi^4 - 48\xi^2 + 12) - 8(8\xi^3 - 12\xi) \\
&= 32\xi^5 - 96\xi^3 - 64\xi^3 + 24\xi + 96\xi \\
&= 32\xi^5 - 160\xi^3 + 120\xi
\end{aligned}$$

Use the recursion relation with $n = 5$ to determine H_6.

$$\begin{aligned}
H_6(\xi) &= 2\xi H_5(\xi) - 2(5)H_4(\xi) \\
&= 2\xi(32\xi^5 - 160\xi^3 + 120\xi) - 10(16\xi^4 - 48\xi^2 + 12) \\
&= 64\xi^6 - 320\xi^4 + 240\xi^2 - 160\xi^4 + 480\xi^2 - 120 \\
&= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120
\end{aligned}$$

Part (c)

Check to see if the differential relation is true by differentiating H_5.

$$\frac{dH_5}{d\xi} \stackrel{?}{=} 2(5)H_4(\xi)$$

$$\frac{d}{d\xi}(32\xi^5 - 160\xi^3 + 120\xi) \stackrel{?}{=} 10(16\xi^4 - 48\xi^2 + 12)$$

$$160\xi^4 - 480\xi^2 + 120 = 160\xi^4 - 480\xi^2 + 120$$

Now check to see if the differential relation is true by differentiating H_6.

$$\frac{dH_6}{d\xi} \stackrel{?}{=} 2(6)H_5(\xi)$$

$$\frac{d}{d\xi}(64\xi^6 - 480\xi^4 + 720\xi^2 - 120) \stackrel{?}{=} 12(32\xi^5 - 160\xi^3 + 120\xi)$$

$$384\xi^5 - 1920\xi^3 + 1440\xi = 384\xi^5 - 1920\xi^3 + 1440\xi$$

Part (d)

The Taylor series expansion of a function $f(z)$ centered at $z = z_0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

If it's centered at $z = 0$, the formula reduces to

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

For $f(z) = e^{-z^2+2z\xi}$ in particular, the zeroth, first, second, and third derivatives are

$$f^{(0)}(z) = e^{-z^2+2z\xi} \quad \rightarrow \quad f^{(0)}(0) = 1$$

$$f^{(1)}(z) = e^{-z^2+2z\xi}(-2z + 2\xi) \quad \rightarrow \quad f^{(1)}(0) = 2\xi$$

$$f^{(2)}(z) = e^{-z^2+2z\xi}(-2z + 2\xi)^2 + e^{-z^2+2z\xi}(-2) \quad \rightarrow \quad f^{(2)}(0) = 4\xi^2 - 2$$

$$f^{(3)}(z) = e^{-z^2+2z\xi}(-2z + 2\xi)^3 + 2e^{-z^2+2z\xi}(-2z + 2\xi)(-2) + e^{-z^2+2z\xi}(4z - 4\xi) \quad \rightarrow \quad f^{(3)}(0) = 8\xi^3 - 12\xi.$$

Therefore, the Taylor series expansion about $z = 0$ is

$$\begin{aligned} e^{-z^2+2z\xi} &= f^{(0)}(0)\frac{z^0}{0!} + f^{(1)}(0)\frac{z^1}{1!} + f^{(2)}(0)\frac{z^2}{2!} + f^{(3)}(0)\frac{z^3}{3!} + \cdots \\ &= 1 + 2\xi z + (4\xi^2 - 2)\frac{z^2}{2} + (8\xi^3 - 12\xi)\frac{z^3}{6} + \cdots . \end{aligned}$$

Notice that $f^{(n)}(0)$ is the nth Hermite polynomial.

$$\begin{aligned} H_0(\xi) &= 1 \\ H_1(\xi) &= 2\xi \\ H_2(\xi) &= 4\xi^2 - 2 \\ H_3(\xi) &= 8\xi^3 - 12\xi \end{aligned}$$