Problem 2.21

The gaussian wave packet. A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2},$$

where A and a are (real and positive) constants.

(a) Normalize $\Psi(x,0)$.

(b) Find $\Psi(x,t)$. *Hint:* Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\,dx$$

can be handled "by completing the square": Let $y \equiv \sqrt{a}\,[x + (b/2a)]$, and note that $\left(ax^2 + bx\right) = y^2 - (b^2/4a)$. *Answer:*

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}, \quad \text{where} \quad \gamma \equiv \sqrt{1 + (2i\hbar at/m)}. \tag{2.111}$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{a/\left[1 + (2\hbar at/m)^2\right]}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$, as time goes on?

(d) Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, σ_x, and σ_p. *Partial answer:* $\langle p^2\rangle = a\hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

Solution

The governing equation for the wave function is the Schrödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

For a free particle in particular, $V = 0$, and the equation simplifies to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}.$$

Let there be a prescribed initial condition, $\Psi(x,0) = \Psi_0(x)$. Since the PDE is linear and $-\infty < x < \infty$, the Fourier transform can be applied to solve it. The Fourier transform of $\Psi(x,t)$ is defined here as

$$\mathcal{F}\{\Psi(x,t)\} = \tilde{\Psi}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\Psi(x,t)\,dx.$$

As a result, the partial derivatives of Ψ transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n \Psi}{\partial x^n}\right\} = (ik)^n \tilde{\Psi}(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n \Psi}{\partial t^n}\right\} = \frac{d^n \tilde{\Psi}}{dt^n}$$

Take the Fourier transform of both sides of the Schrödinger equation.

$$\mathcal{F}\left\{i\hbar\frac{\partial \Psi}{\partial t}\right\} = \mathcal{F}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\right\}$$

Since the transform is a linear operator, the constants can be brought in front.

$$i\hbar\mathcal{F}\left\{\frac{\partial \Psi}{\partial t}\right\} = -\frac{\hbar^2}{2m}\mathcal{F}\left\{\frac{\partial^2 \Psi}{\partial x^2}\right\}$$

Transform the derivatives.

$$i\hbar\frac{d\tilde{\Psi}}{dt} = -\frac{\hbar^2}{2m}(ik)^2\tilde{\Psi}(k,t)$$

By using the Fourier transform, Schrödinger's equation has become a first-order ODE. Solve it now.

$$\frac{\frac{d\tilde{\Psi}}{dt}}{\tilde{\Psi}} = -\frac{i\hbar}{2m}k^2$$
$$\frac{d}{dt}\ln\tilde{\Psi} = -\frac{i\hbar}{2m}k^2$$
$$\ln\tilde{\Psi} = -\frac{i\hbar}{2m}k^2 t + C(k)$$
$$\tilde{\Psi}(k,t) = B(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right)$$

To determine $B(k)$, transform the given initial condition,

$$\Psi(x,0) = \Psi_0(x) \quad \rightarrow \quad \mathcal{F}\{\Psi(x,0)\} = \mathcal{F}\{\Psi_0(x)\} \quad \rightarrow \quad \tilde{\Psi}(k,0) = \tilde{\Psi}_0(k),$$

and then apply it.

$$\tilde{\Psi}(k,0) = B(k) = \tilde{\Psi}_0(k)$$

Consequently, the transformed solution to Schrödinger's equation is

$$\tilde{\Psi}(k,t) = \tilde{\Psi}_0(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right).$$

Now take the inverse Fourier transform of both sides to get $\Psi(x,t)$.

$$\Psi(x,t) = \mathcal{F}^{-1}\left\{\tilde{\Psi}(k,t)\right\} = \mathcal{F}^{-1}\left\{\tilde{\Psi}_0(k)\exp\left(-\frac{i\hbar}{2m}k^2 t\right)\right\}$$

According to the convolution theorem for the Fourier transform,

$$\mathcal{F}^{-1}\left\{\tilde{f}(k)\tilde{g}(k)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

The inverse Fourier transform of $\tilde{\Psi}_0(k)$ is just $\Psi_0(x)$, and the inverse Fourier transform of the exponential function is

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\exp\left(-\frac{i\hbar}{2m}k^2 t\right)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\exp\left(-\frac{i\hbar}{2m}k^2 t\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{i\hbar}{2m}k^2 t + ikx\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar}{2m}t\left(k^2 - \frac{2mx}{\hbar t}k\right)\right]dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar}{2m}t\left(k^2 - \frac{2mx}{\hbar t}k + \frac{m^2x^2}{\hbar^2t^2}\right) + \frac{i\hbar}{2m}t\left(\frac{m^2x^2}{\hbar^2t^2}\right)\right]dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar t}{2m}\left(k - \frac{mx}{\hbar t}\right)^2\right]\exp\left(\frac{imx^2}{2\hbar t}\right)dk \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{i\hbar t}{2m}u^2\right)du \\
&= \frac{2}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\int_0^{\infty}\exp\left[-\frac{u^2}{\left(\sqrt{\frac{2m}{i\hbar t}}\right)^2}\right]du \\
&= \frac{2}{\sqrt{2\pi}}\exp\left(\frac{imx^2}{2\hbar t}\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{2m}{i\hbar t}}}{2}\right) \\
&= \sqrt{\frac{m}{i\hbar t}}\exp\left(\frac{imx^2}{2\hbar t}\right).
\end{aligned}$$

By the convolution theorem, then,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_0(\xi)\sqrt{\frac{m}{i\hbar t}}\exp\left[\frac{im(x-\xi)^2}{2\hbar t}\right]d\xi.$$

Therefore, the solution to the Schrödinger equation for a free particle is

$$\boxed{\Psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right]\Psi_0(\xi)\,d\xi.}$$

In this problem

$$\Psi(x,0) = \Psi_0(x) = Ae^{-ax^2}.$$

Part (a)

Start by normalizing the initial wave function: Require that the integral of $|\Psi(x,0)|^2$ over the whole line is 1 in order to determine A.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx \\
&= \int_{-\infty}^{\infty} A^2 e^{-2ax^2}\,dx \\
&= 2A^2 \int_0^{\infty} \exp\left[-\frac{x^2}{\left(\frac{1}{\sqrt{2a}}\right)^2}\right] dx \\
&= 2A^2 \cdot \sqrt{\pi}\left(\frac{\frac{1}{\sqrt{2a}}}{2}\right) \\
&= A^2\sqrt{\frac{\pi}{2a}}
\end{aligned}$$

Solve for A.

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

As a result, the initial wave function is

$$\Psi_0(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}.$$

Part (b)

Substitute the initial wave function into the boxed formula for $\Psi(x,t)$.

$$\begin{aligned}
\Psi(x,t) &= \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] \Psi_0(\xi)\,d\xi \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] \left(\frac{2a}{\pi}\right)^{1/4} e^{-a\xi^2}\,d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2 - a\xi^2\right] d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar t}x^2 - \frac{im}{\hbar t}x\xi + \frac{im - 2\hbar at}{2\hbar t}\xi^2\right) d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar t}x^2\right) \exp\left[\frac{im - 2\hbar at}{2\hbar t}\left(\xi^2 - \frac{2imx}{im - 2\hbar at}\xi\right)\right] d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im}{2\hbar t}x^2\right) \int_{-\infty}^{\infty} \exp\left[\frac{im - 2\hbar at}{2\hbar t}\left(\xi - \frac{imx}{im - 2\hbar at}\right)^2 - \frac{im - 2\hbar at}{2\hbar t}\frac{i^2m^2x^2}{(im - 2\hbar at)^2}\right] d\xi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im}{2\hbar t}x^2\right) \int_{-\infty}^{\infty} \exp\left[\frac{im - 2\hbar at}{2\hbar t}\left(\xi - \frac{imx}{im - 2\hbar at}\right)^2\right] \exp\left[\frac{m^2x^2}{2\hbar t(im - 2\hbar at)}\right] d\xi
\end{aligned}$$

Make the following substitution.

$$u = \xi - \frac{imx}{im - 2\hbar at}$$
$$du = d\xi$$

Therefore,

$$\begin{aligned}
\Psi(x,t) &= \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im}{2\hbar t}x^2 + \frac{m^2x^2}{2\hbar t(im-2\hbar at)}\right]\int_{-\infty}^{\infty}\exp\left(\frac{im-2\hbar at}{2\hbar t}u^2\right)du \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im(im-2\hbar at)+m^2}{2\hbar t(im-2\hbar at)}x^2\right]\int_{-\infty}^{\infty}\exp\left[-\frac{u^2}{\left(\sqrt{\frac{2\hbar t}{2\hbar at - im}}\right)^2}\right]du \\
&= 2\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{-2i\hbar amt}{2\hbar t(im-2\hbar at)}x^2\right]\int_{0}^{\infty}\exp\left[-\frac{u^2}{\left(\sqrt{\frac{2\hbar t}{2\hbar at - im}}\right)^2}\right]du \\
&= 2\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left(-\frac{iam}{im-2\hbar at}x^2\right)\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{2\hbar t}{2\hbar at - im}}}{2}\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{i(2\hbar at - im)}}\exp\left(-\frac{a}{1+\frac{2i\hbar at}{m}}x^2\right) \\
&= \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right),
\end{aligned}$$

or if we set $\gamma = \sqrt{1 + (2i\hbar at/m)}$,

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\gamma}e^{-ax^2/\gamma^2}.$$

Part (c)

Now that the wave function is known, the probability distribution for the particle's position at time t can be determined.

$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\
&= \left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)\right]\left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\right] \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}} - \frac{ax^2}{1-\frac{2i\hbar at}{m}}\right) \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[\frac{-\left(1-\frac{2i\hbar at}{m}\right)ax^2 - \left(1+\frac{2i\hbar at}{m}\right)ax^2}{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}\right]
\end{aligned}$$

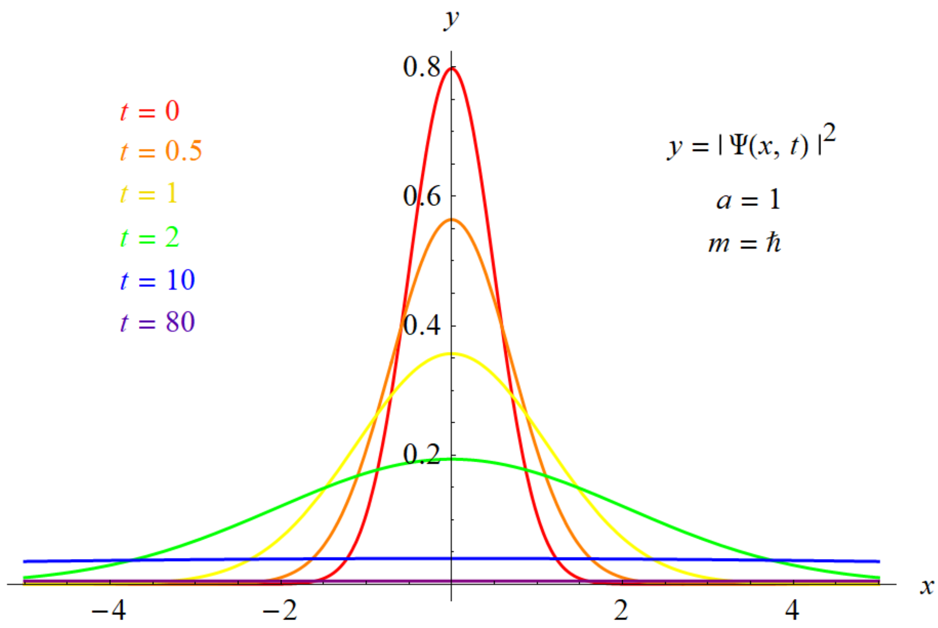
Therefore,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right),$$

or if we set $w = \sqrt{a/\left[1+(2\hbar at/m)^2\right]}$,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\,we^{-2w^2x^2}.$$

Below is a plot of the probability distribution versus x at several times for the special case that $a = 1$ and $m = \hbar$.



Part (d)

Now that the probability distribution for the particle's position at time t is known, expectation values can be determined. Calculate the expectation value of x at time t.

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty}x\sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty}x\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= 0
\end{aligned}$$

This integral is zero because the integrand is an odd function and it's taken over a symmetric interval.

Calculate the expectation value of x^2 at time t.

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x^2\sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= \sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{-\infty}^{\infty} x^2\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= 2\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\int_{0}^{\infty} x^2\exp\left[-\frac{x^2}{\left(\sqrt{\frac{1}{2a}+\frac{2\hbar^2at^2}{m^2}}\right)^2}\right]dx \\
&= 2\sqrt{\frac{2m^2a}{\pi(m^2+4\hbar^2a^2t^2)}}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)^3 \\
&= \frac{m^2+4\hbar^2a^2t^2}{4m^2a}
\end{aligned}$$

The standard deviation in x at time t is then

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}.$$

Calculate the expectation value of p at time t.

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial\Psi}{\partial x}\,dx \\
&= -i\hbar\int_{-\infty}^{\infty}\left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\right]\frac{\partial}{\partial x}\left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)\right]dx \\
&= -i\hbar\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)}}\int_{-\infty}^{\infty}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\frac{\partial}{\partial x}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)dx \\
&= -i\hbar\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)\left(-\frac{2ax}{1+\frac{2i\hbar at}{m}}\right)dx \\
&= i\hbar\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\frac{2a}{1+\frac{2i\hbar at}{m}}\int_{-\infty}^{\infty}x\exp\left(\frac{-2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= 0
\end{aligned}$$

This integral is zero because the integrand is an odd function and it's taken over a symmetric interval. This result could have been obtained from Ehrenfest's theorem: $\langle p\rangle = m\,d\langle x\rangle/dt = 0$.

Calculate the expectation value of p^2 at time t.

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi(x,t)\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial^2\Psi}{\partial x^2}\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty} \left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\right]\frac{\partial^2}{\partial x^2}\left[\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)\right]dx \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1-\frac{2i\hbar at}{m}\right)\left(1+\frac{2i\hbar at}{m}\right)}}\int_{-\infty}^{\infty}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\frac{\partial^2}{\partial x^2}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)dx \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\frac{\partial}{\partial x}\left[\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)\left(-\frac{2ax}{1+\frac{2i\hbar at}{m}}\right)\right]dx \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{ax^2}{1-\frac{2i\hbar at}{m}}\right)\left[\left(-\frac{2ax}{1+\frac{2i\hbar at}{m}}\right)^2+\left(-\frac{2a}{1+\frac{2i\hbar at}{m}}\right)\right]\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)dx \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\left[\left(-\frac{2a}{1+\frac{2i\hbar at}{m}}\right)^2x^2+\left(-\frac{2a}{1+\frac{2i\hbar at}{m}}\right)\right]\exp\left(-\frac{2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left[\left(-\frac{2a}{1+\frac{2i\hbar at}{m}}\right)^2\int_{-\infty}^{\infty}x^2\exp\left(-\frac{2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx\right. \\
&\qquad\left.-\frac{2a}{1+\frac{2i\hbar at}{m}}\int_{-\infty}^{\infty}\exp\left(-\frac{2ax^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dx\right] \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left\{\frac{8a^2}{1+\frac{4i\hbar at}{m}-\frac{4\hbar^2a^2t^2}{m^2}}\int_0^{\infty}x^2\exp\left[-\frac{x^2}{\left(\sqrt{\frac{1}{2a}+\frac{2\hbar^2at^2}{m^2}}\right)^2}\right]dx\right. \\
&\qquad\left.-\frac{4a}{1+\frac{2i\hbar at}{m}}\int_0^{\infty}\exp\left[-\frac{x^2}{\left(\sqrt{\frac{1}{2a}+\frac{2\hbar^2at^2}{m^2}}\right)^2}\right]dx\right\} \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left[\frac{8a^2}{1+\frac{4i\hbar at}{m}-\frac{4\hbar^2a^2t^2}{m^2}}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)^3\right. \\
&\qquad\left.-\frac{4a}{1+\frac{2i\hbar at}{m}}\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{2m^2a}}}{2}\right)\right]
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\langle p^2\rangle &= -\hbar^2\sqrt{\frac{2a}{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left(\frac{2a^2}{1+\frac{4i\hbar at}{m}-\frac{4\hbar^2a^2t^2}{m^2}}\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}\frac{m^2+4\hbar^2a^2t^2}{2m^2a}\right.\\
&\qquad\left.-\frac{2a}{1+\frac{2i\hbar at}{m}}\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}\right)\\
&= -\hbar^2\left[\frac{a(m^2+4\hbar^2a^2t^2)}{m^2+4i\hbar atm-4\hbar^2a^2t^2}-\frac{2ma}{m+2i\hbar at}\right]\\
&= -\hbar^2\left[\frac{a(m^2+4\hbar^2a^2t^2)(m+2i\hbar at)-2ma(m^2+4i\hbar atm-4\hbar^2a^2t^2)}{(m^2+4i\hbar atm-4\hbar^2a^2t^2)(m+2i\hbar at)}\right]\\
&= -\hbar^2\left(\frac{-am^3-6i\hbar m^2a^2t+12\hbar^2ma^3t^2+8i\hbar^3a^4t^3}{m^3+6i\hbar m^2at-12\hbar^2ma^2t^2-8i\hbar^3a^3t^3}\right)\\
&= -\hbar^2\left[\frac{-a(m+2i\hbar at)^3}{(m+2i\hbar at)^3}\right]\\
&= -\hbar^2(-a)\\
&= a\hbar^2
\end{aligned}$$

The standard deviation in p at time t is then

$$\sigma_p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{a\hbar^2}=\hbar\sqrt{a}.$$

Part (e)

The uncertainty product is

$$\sigma_x\sigma_p=\sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}(\hbar\sqrt{a})=\frac{\hbar}{2}\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}},$$

which is consistent with Heisenberg's uncertainty principle ($\sigma_x\sigma_p \geq \hbar/2$) for all t. This system comes closest to the limit at $t=0$.