Problem 2.23

Delta functions live under integral signs, and two expressions ($D_1(x)$ and $D_2(x)$) involving delta functions are said to be equal if

$$\int_{-\infty}^{+\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{+\infty} f(x)D_2(x)\,dx,$$

for every (ordinary) function $f(x)$.

(a) Show that

$$\delta(cx) = \frac{1}{|c|}\delta(x), \tag{2.145}$$

where c is a real constant. (Be sure to check the case where c is negative.)

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{2.146}$$

(In the rare case where it actually matters, we define $\theta(0)$ to be 1/2.) Show that $d\theta/dx = \delta(x)$.

Solution

Part (a)

Consider the integral,

$$I = \int_{-\infty}^{\infty} f(x)\delta(cx)\,dx.$$

Assuming that c is positive, make the following substitution.

$$u = cx \quad \rightarrow \quad x = \frac{u}{c}$$

$$du = c\,dx \quad \rightarrow \quad dx = \frac{du}{c}$$

Consequently,

$$I = \int_{c\times-\infty}^{c\times\infty} f\left(\frac{u}{c}\right)\delta(u)\left(\frac{du}{c}\right) = \frac{1}{c}\int_{-\infty}^{\infty} f\left(\frac{u}{c}\right)\delta(u)\,du = \frac{1}{c}f(0).$$

Make the same substitution again, this time assuming that c is negative.

$$I = \int_{c\times-\infty}^{c\times\infty} f\left(\frac{u}{c}\right)\delta(u)\left(\frac{du}{c}\right) = \frac{1}{c}\int_{\infty}^{-\infty} f\left(\frac{u}{c}\right)\delta(u)\,du = -\frac{1}{c}\int_{-\infty}^{\infty} f\left(\frac{u}{c}\right)\delta(u)\,du = -\frac{1}{c}f(0)$$

Now combine these results.

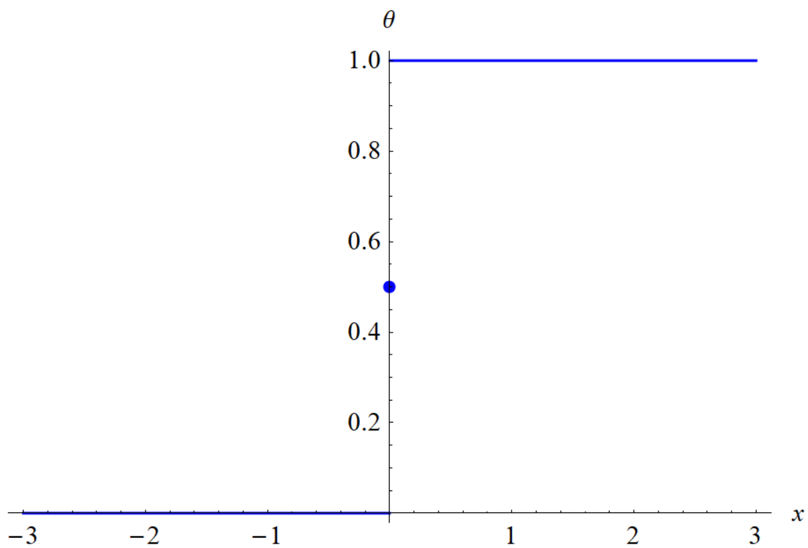
$$I = \frac{1}{|c|}f(0) = \frac{1}{|c|}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(x)\frac{\delta(x)}{|c|}\,dx$$

Therefore,

$$\delta(cx) = \frac{\delta(x)}{|c|}.$$

Part (b)

Below is a plot of the Heaviside function versus x.



It's clear from the graph that

$$\frac{d\theta}{dx} = \begin{cases} 0 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}.$$

All that remains to show is that $d\theta/dx$ behaves the same way in the integrand as the delta function does. Suppose that a and b are positive real numbers.

$$\int_{-a}^{b} f(x)\delta(x)\,dx = f(0)$$

Replace $\delta(x)$ with $d\theta/dx$ and evaluate the integral.

$$\begin{aligned}
\int_{-a}^{b} f(x)\frac{d\theta}{dx}\,dx &= f(x)\theta(x)\Big|_{-a}^{b} - \int_{-a}^{b} \frac{df}{dx}\theta(x)\,dx \\
&= f(b)\theta(b) - f(-a)\theta(-a) - \left[\int_{-a}^{0} \frac{df}{dx}(0)\,dx + \int_{0}^{b} \frac{df}{dx}(1)\,dx\right] \\
&= f(b)(1) - f(-a)(0) - \int_{0}^{b} \frac{df}{dx}\,dx \\
&= f(b) - f(x)\Big|_{0}^{b} \\
&= f(b) - [f(b) - f(0)] \\
&= f(0)
\end{aligned}$$

Therefore,

$$\frac{d\theta}{dx} = \delta(x).$$