Problem 2.25

Check that the bound state of the delta-function well (Equation 2.132) is orthogonal to the scattering states (Equations 2.134 and 2.135).

Solution

The bound state of the delta-function well is in Equation 2.132,

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2},$$

whereas the scattering states are in Equations 2.134 and 2.135,

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < 0 \\ Fe^{ikx} + Ge^{-ikx} & \text{if } x > 0 \end{cases}.$$

Check to see that these two eigenstates are orthogonal.

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx &= \int_{-\infty}^{0} \psi_m^* \psi_n \, dx + \int_{0}^{\infty} \psi_m^* \psi_n \, dx \\
&= \int_{-\infty}^{0} \left(\frac{\sqrt{m\alpha}}{\hbar} e^{m\alpha x/\hbar^2} \right)^* \left(Ae^{ikx} + Be^{-ikx} \right) dx \\
&\qquad + \int_{0}^{\infty} \left(\frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha x/\hbar^2} \right)^* \left(Fe^{ikx} + Ge^{-ikx} \right) dx \\
&= \int_{-\infty}^{0} \left(\frac{\sqrt{m\alpha}}{\hbar} e^{m\alpha x/\hbar^2} \right) \left(Ae^{ikx} + Be^{-ikx} \right) dx \\
&\qquad + \int_{0}^{\infty} \left(\frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha x/\hbar^2} \right) \left(Fe^{ikx} + Ge^{-ikx} \right) dx \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left(A \int_{-\infty}^{0} e^{m\alpha x/\hbar^2} e^{ikx} \, dx + B \int_{-\infty}^{0} e^{m\alpha x/\hbar^2} e^{-ikx} \, dx \right. \\
&\qquad \left. + F \int_{0}^{\infty} e^{-m\alpha x/\hbar^2} e^{ikx} \, dx + G \int_{0}^{\infty} e^{-m\alpha x/\hbar^2} e^{-ikx} \, dx \right) \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left(\frac{A}{\frac{m\alpha}{\hbar^2} + ik} e^{(m\alpha/\hbar^2 + ik)x} \Big|_{-\infty}^{0} + \frac{B}{\frac{m\alpha}{\hbar^2} - ik} e^{(m\alpha/\hbar^2 - ik)x} \Big|_{-\infty}^{0} \right. \\
&\qquad \left. + \frac{F}{-\frac{m\alpha}{\hbar^2} + ik} e^{(-m\alpha/\hbar^2 + ik)x} \Big|_{0}^{\infty} + \frac{G}{-\frac{m\alpha}{\hbar^2} - ik} e^{(-m\alpha/\hbar^2 - ik)x} \Big|_{0}^{\infty} \right) \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left(\frac{A}{\frac{m\alpha}{\hbar^2} + ik} + \frac{B}{\frac{m\alpha}{\hbar^2} - ik} - \frac{F}{-\frac{m\alpha}{\hbar^2} + ik} - \frac{G}{-\frac{m\alpha}{\hbar^2} - ik} \right) \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left(\frac{A + G}{\frac{m\alpha}{\hbar^2} + ik} + \frac{B + F}{\frac{m\alpha}{\hbar^2} - ik} \right) \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left[\frac{(A + G)\left(\frac{m\alpha}{\hbar^2} - ik\right) + (B + F)\left(\frac{m\alpha}{\hbar^2} + ik\right)}{\frac{m^2\alpha^2}{\hbar^4} + k^2} \right]
\end{aligned}$$

Organize the terms in the numerator.

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \frac{\sqrt{m\alpha}}{\hbar} \left[\frac{\frac{m\alpha}{\hbar^2}(A+B+F+G) + ik(B+F-A-G)}{\frac{m^2\alpha^2}{\hbar^4} + k^2} \right]$$

Apply the boundary conditions of the delta-function well here (the continuity of ψ at $x = 0$ and the discontinuity of $d\psi/dx$ at $x = 0$).

$$F + G = A + B$$
$$ik(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx &= \frac{\sqrt{m\alpha}}{\hbar} \left\{ \frac{\frac{m\alpha}{\hbar^2}[A+B+(A+B)] + \left[-\frac{2m\alpha}{\hbar^2}(A+B)\right]}{\frac{m^2\alpha^2}{\hbar^4} + k^2} \right\} \\
&= \frac{\sqrt{m\alpha}}{\hbar} \left[\frac{\frac{2m\alpha}{\hbar^2}(A+B) - \frac{2m\alpha}{\hbar^2}(A+B)}{\frac{m^2\alpha^2}{\hbar^4} + k^2} \right] \\
&= 0,
\end{aligned}$$

which means the bound and scattering eigenstates of the delta-function well are orthogonal.