Problem 2.26

What is the Fourier transform of $\delta(x)$? Using Plancherel's theorem, show that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk. \tag{2.147}$$

Comment: This formula gives any respectable mathematician apoplexy. Although the integral is clearly infinite when $x = 0$, it doesn't converge (to zero or anything else) when $x \neq 0$, since the integrand oscillates forever. There are ways to patch it up (for instance, you can integrate from $-L$ to $+L$, and interpret Equation 2.147 to mean the *average* value of the finite integral, as $L \to \infty$). The source of the problem is that the delta function doesn't meet the requirement (square-integrability) for Plancherel's theorem (see footnote 42). In spite of this, Equation 2.147 can be extremely useful, if handled with care.

Solution

The Fourier transform of a function is defined as

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx,$$

so

$$\begin{aligned}\mathcal{F}\{\delta(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-ik(0)} \\ &= \frac{1}{\sqrt{2\pi}}.\end{aligned}$$

Take the inverse Fourier transform of both sides to get back $\delta(x)$.

$$\begin{aligned}\mathcal{F}^{-1}\left\{\mathcal{F}\{\delta(x)\}\right\} &= \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} \\ \delta(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left(\frac{1}{\sqrt{2\pi}}\right) dk\end{aligned}$$

Therefore, pulling the constants in front,

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk.$$