Problem 2.28

Find the transmission coefficient, for the potential in Problem 2.27.

Solution

The governing equation for the wave function $\Psi(x,t)$ is the Schrödinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

If $V(x,t) = -\alpha[\delta(x+a) + \delta(x-a)]$, then it reduces to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha[\delta(x+a) + \delta(x-a)]\Psi(x,t),$$

which can be solved by the method of separation of variables because the PDE and its associated boundary conditions (Ψ and its derivatives go to zero as $x \to \pm\infty$) are linear and homogeneous. Since the eigenstates and their energies are of interest, this method is opted for. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] - \alpha[\delta(x+a) + \delta(x-a)][\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) - \alpha[\delta(x+a) + \delta(x-a)]\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ in order to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a) + \delta(x-a)]$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a) + \delta(x-a)] = E$$

As a result of using the method of separation of variables, the Schrödinger equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha[\delta(x+a) + \delta(x-a)] &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a) + \delta(x-a)] + E\right\}\psi(x) = 0. \tag{1}$$

If $x \neq -a, a$, then it simplifies to

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0, \quad x \neq -a, a.$$

Scattering states have energy $E > 0$; in this case, the general solution is

$$\psi(x) = \begin{cases} C_1 \exp\left(i\frac{\sqrt{2mE}}{\hbar}x\right) + C_2 \exp\left(-i\frac{\sqrt{2mE}}{\hbar}x\right) & \text{if } x < -a \\ C_3 \exp\left(i\frac{\sqrt{2mE}}{\hbar}x\right) + C_4 \exp\left(-i\frac{\sqrt{2mE}}{\hbar}x\right) & \text{if } -a < x < a \\ C_5 \exp\left(i\frac{\sqrt{2mE}}{\hbar}x\right) + C_6 \exp\left(-i\frac{\sqrt{2mE}}{\hbar}x\right) & \text{if } x > a \end{cases}.$$

Introduce the constant,

$$k = \frac{\sqrt{2mE}}{\hbar},$$

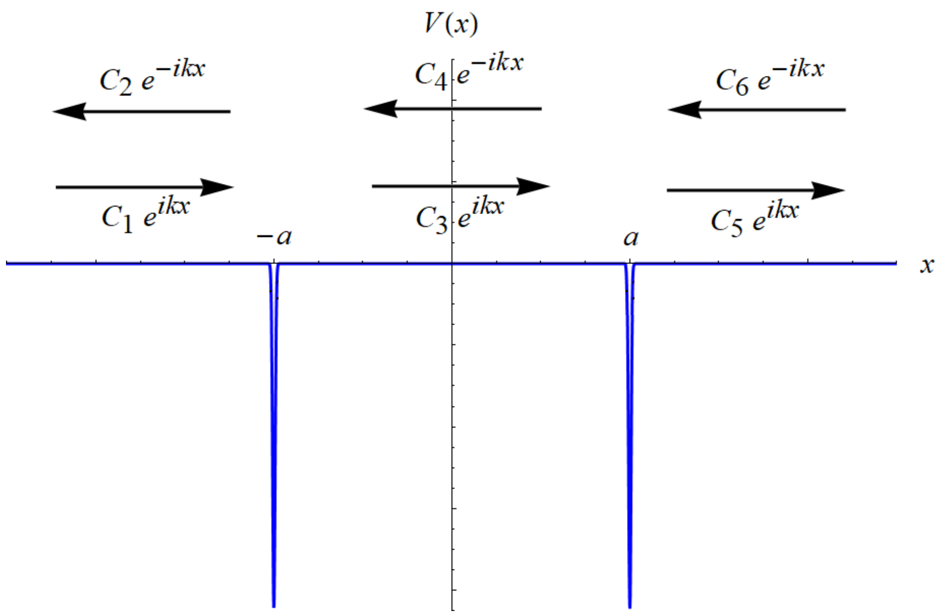
to make things more manageable.

$$\psi(x) = \begin{cases} C_1 e^{ikx} + C_2 e^{-ikx} & \text{if } x < -a \\ C_3 e^{ikx} + C_4 e^{-ikx} & \text{if } -a < x < a \\ C_5 e^{ikx} + C_6 e^{-ikx} & \text{if } x > a \end{cases}.$$

Since $\phi(t) = e^{-iEt/\hbar}$, the product solution is

$$\psi(x)\phi(t) = \begin{cases} C_1 e^{i(kx - Et/\hbar)} + C_2 e^{-i(kx+Et/\hbar)} & \text{if } x < -a \\ C_3 e^{i(kx - Et/\hbar)} + C_4 e^{-i(kx+Et/\hbar)} & \text{if } -a < x < a \\ C_5 e^{i(kx - Et/\hbar)} + C_6 e^{-i(kx+Et/\hbar)} & \text{if } x > a \end{cases},$$

which is a linear combination of waves travelling to the right and to the left with speed $E/(\hbar k) = \sqrt{E/(2m)}$. Assuming a plane wave is coming from the left, $C_6 = 0$, the transmission coefficient is $|C_5/C_1|^2$, and the reflection coefficient is $|C_2/C_1|^2$.



Use the fact that the wave function [and consequently $\psi(x)$] is continuous across $x=-a$ and $x=a$ to obtain two conditions.

$$\lim_{x\to -a^-}\psi(x) = \lim_{x\to -a^+}\psi(x): \quad C_1e^{-ika}+C_2e^{ika} = C_3e^{-ika}+C_4e^{ika}$$

$$\lim_{x\to +a^-}\psi(x) = \lim_{x\to +a^+}\psi(x): \quad C_3e^{ika}+C_4e^{-ika} = C_5e^{ika}+\cancel{C_6e^{-ika}}$$

Another condition can be obtained by integrating both sides of equation (1) with respect to x from $-a-\epsilon$ to $-a+\epsilon$, where ϵ is a really small positive number.

$$\int_{-a-\epsilon}^{-a+\epsilon}\left\{\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a)+\delta(x-a)]+E\right\}\psi(x)\right\}dx = \int_{-a-\epsilon}^{-a+\epsilon}0\,dx$$

$$\int_{-a-\epsilon}^{-a+\epsilon}\frac{d^2\psi}{dx^2}\,dx+\frac{2m}{\hbar^2}\left\{\alpha\int_{-a-\epsilon}^{-a+\epsilon}[\delta(x+a)+\delta(x-a)]\psi(x)\,dx+E\int_{-a-\epsilon}^{-a+\epsilon}\psi(x)\,dx\right\} = 0$$

$$\left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon}+\frac{2m}{\hbar^2}\left[\alpha\psi(-a)+E\psi(-a)\int_{-a-\epsilon}^{-a+\epsilon}dx\right] = 0$$

$$\left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon}+\frac{2m}{\hbar^2}\left[\alpha\psi(-a)+E\psi(-a)(2\epsilon)\right] = 0$$

Take the limit as $\epsilon\to 0$.

$$\left.\frac{d\psi}{dx}\right|_{-a^-}^{-a^+}+\frac{2m\alpha}{\hbar^2}\psi(-a) = 0$$

$$\frac{2m\alpha}{\hbar^2}\psi(-a) = \lim_{x\to -a^-}\frac{d\psi}{dx}-\lim_{x\to -a^+}\frac{d\psi}{dx}$$

$$\frac{2m\alpha}{\hbar^2}\left(C_1e^{-ika}+C_2e^{ika}\right) = \left(iC_1ke^{-ika}-iC_2ke^{ika}\right)-\left(iC_3ke^{-ika}-iC_4ke^{ika}\right)$$

The final condition is obtained by integrating both sides of equation (1) with respect to x from $a-\epsilon$ to $a+\epsilon$.

$$\int_{a-\epsilon}^{a+\epsilon}\left\{\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a)+\delta(x-a)]+E\right\}\psi(x)\right\}dx = \int_{a-\epsilon}^{a+\epsilon}0\,dx$$

$$\int_{a-\epsilon}^{a+\epsilon}\frac{d^2\psi}{dx^2}\,dx+\frac{2m}{\hbar^2}\left\{\alpha\int_{a-\epsilon}^{a+\epsilon}[\delta(x+a)+\delta(x-a)]\psi(x)\,dx+E\int_{a-\epsilon}^{a+\epsilon}\psi(x)\,dx\right\} = 0$$

$$\left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon}+\frac{2m}{\hbar^2}\left[\alpha\psi(a)+E\psi(a)\int_{a-\epsilon}^{a+\epsilon}dx\right] = 0$$

$$\left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon}+\frac{2m}{\hbar^2}\left[\alpha\psi(a)+E\psi(a)(2\epsilon)\right] = 0$$

Take the limit as $\epsilon\to 0$.

$$\left.\frac{d\psi}{dx}\right|_{a^-}^{a^+}+\frac{2m\alpha}{\hbar^2}\psi(a) = 0$$

$$\frac{2m\alpha}{\hbar^2}\psi(a) = \lim_{x\to a^-}\frac{d\psi}{dx}-\lim_{x\to a^+}\frac{d\psi}{dx}$$

$$\frac{2m\alpha}{\hbar^2}\left(C_5e^{ika}+\cancel{C_6e^{-ika}}\right) = \left(iC_3ke^{ika}-iC_4ke^{-ika}\right)-\left(iC_5ke^{ika}-\cancel{iC_6ke^{-ika}}\right)$$

To summarize, the four equations involving C_1, C_2, C_3, C_4, and C_5 are

$$\begin{cases} C_1e^{-ika} + C_2e^{ika} = C_3e^{-ika} + C_4e^{ika} \\ C_3e^{ika} + C_4e^{-ika} = C_5e^{ika} \\ \dfrac{2m\alpha}{\hbar^2}\left(C_1e^{-ika} + C_2e^{ika}\right) = \left(iC_1ke^{-ika} - iC_2ke^{ika}\right) - \left(iC_3ke^{-ika} - iC_4ke^{ika}\right) \\ \dfrac{2m\alpha}{\hbar^2}C_5e^{ika} = \left(iC_3ke^{ika} - iC_4ke^{-ika}\right) - \left(iC_5ke^{ika}\right) \end{cases}.$$

Solve this first equation for C_4

$$C_4 = C_1e^{-2ika} + C_2 - C_3e^{-2ika}$$

and then substitute the formula into the other three equations.

$$\begin{cases} C_3e^{ika} + \left(C_1e^{-2ika} + C_2 - C_3e^{-2ika}\right)e^{-ika} = C_5e^{ika} \\ \dfrac{2m\alpha}{\hbar^2}\left(C_1e^{-ika} + C_2e^{ika}\right) = iC_1ke^{-ika} - \cancel{iC_2ke^{ika}} - iC_3ke^{-ika} + ik\left(C_1e^{-ika} + \cancel{C_2e^{ika}} - C_3e^{-ika}\right) \\ \dfrac{2m\alpha}{\hbar^2}C_5e^{ika} = iC_3ke^{ika} - ik\left(C_1e^{-2ika} + C_2 - C_3e^{-2ika}\right)e^{-ika} - iC_5ke^{ika} \end{cases}$$

$$\begin{cases} C_1e^{-3ika} + C_2e^{-ika} + C_3(e^{ika} - e^{-3ika}) = C_5e^{ika} \\ \dfrac{2m\alpha}{\hbar^2}\left(C_1e^{-ika} + C_2e^{ika}\right) = 2iC_1ke^{-ika} - 2iC_3ke^{-ika} \\ \left(\dfrac{2m\alpha}{\hbar^2} + ik\right)C_5e^{ika} = -iC_1ke^{-3ika} - iC_2ke^{-ika} + iC_3k(e^{ika} + e^{-3ika}) \end{cases}$$

Solve this first equation for C_3

$$C_3 = \frac{C_5e^{ika} - C_1e^{-3ika} - C_2e^{-ika}}{e^{ika} - e^{-3ika}}$$

and then plug it into the other two equations.

$$\begin{cases} \dfrac{2m\alpha}{\hbar^2}\left(C_1e^{-ika} + C_2e^{ika}\right) = 2iC_1ke^{-ika} - 2i\dfrac{C_5e^{ika} - C_1e^{-3ika} - C_2e^{-ika}}{e^{ika} - e^{-3ika}}ke^{-ika} \\ \left(\dfrac{2m\alpha}{\hbar^2} + ik\right)C_5e^{ika} = -iC_1ke^{-3ika} - iC_2ke^{-ika} + i\dfrac{C_5e^{ika} - C_1e^{-3ika} - C_2e^{-ika}}{e^{ika} - e^{-3ika}}k(e^{ika} + e^{-3ika}) \end{cases}$$

Solve this first equation for C_2

$$C_2 = \frac{C_1m\alpha + e^{4iak}[i(C_1 - C_5)k\hbar^2 - C_1m\alpha]}{(e^{4iak} - 1)m\alpha - ik\hbar^2}e^{-2iak}$$

and then plug it into the last equation.

$$\begin{aligned} \left(\frac{2m\alpha}{\hbar^2} + ik\right)C_5e^{ika} &= -iC_1ke^{-3ika} - i\left\{\frac{C_1m\alpha + e^{4iak}[i(C_1 - C_5)k\hbar^2 - C_1m\alpha]}{(e^{4iak} - 1)m\alpha - ik\hbar^2}e^{-2iak}\right\}ke^{-ika} \\ &\quad + i\frac{C_5e^{ika} - C_1e^{-3ika} - \left\{\frac{C_1m\alpha + e^{4iak}[i(C_1 - C_5)k\hbar^2 - C_1m\alpha]}{(e^{4iak} - 1)m\alpha - ik\hbar^2}e^{-2iak}\right\}e^{-ika}}{e^{ika} - e^{-3ika}}k(e^{ika} + e^{-3ika}) \end{aligned}$$

Solve this equation for C_5.

$$C_5 = \frac{k^2\hbar^4}{m^2\alpha^2 e^{4iak} - (m\alpha + ik\hbar^2)^2} C_1$$

Divide both sides by C_1.

$$\begin{aligned}
\frac{C_5}{C_1} &= \frac{k^2\hbar^4}{m^2\alpha^2 e^{4iak} - (m\alpha + ik\hbar^2)^2} \\
&= \frac{k^2\hbar^4}{m^2\alpha^2(\cos 4ak + i\sin 4ak) - (m^2\alpha^2 + 2ikm\alpha\hbar^2 - k^2\hbar^4)} \\
&= \frac{k^2\hbar^4}{k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak) + i(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)} \\
&= \frac{k^2\hbar^4}{k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak) + i(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)} \times \frac{k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak) - i(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)}{k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak) - i(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)} \\
&= \frac{k^2\hbar^4[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)] - ik^2\hbar^4(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)}{[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)]^2 + (m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)^2}
\end{aligned}$$

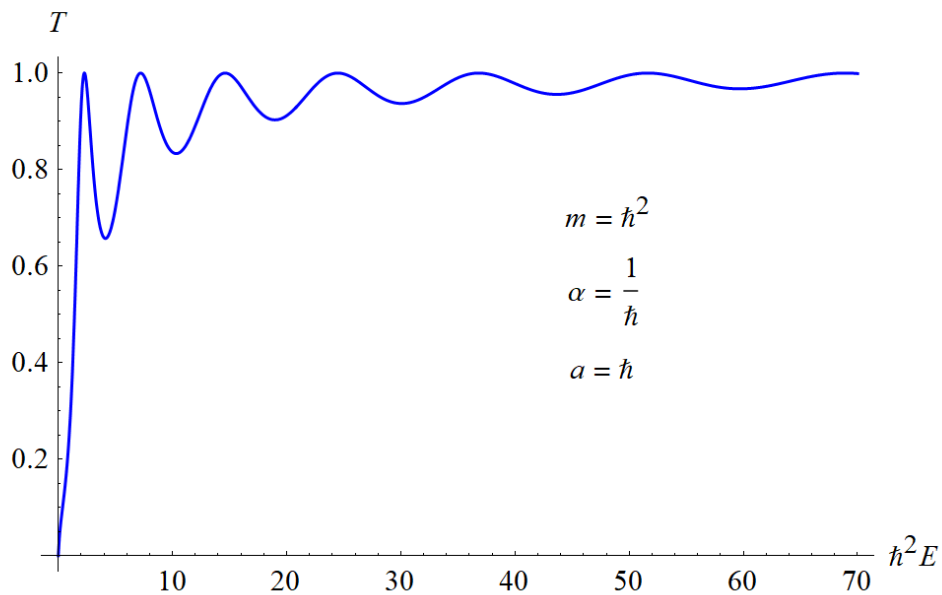
Multiply both sides by the complex conjugate to get the transmission coefficient.

$$\begin{aligned}
\left|\frac{C_5}{C_1}\right|^2 &= \frac{k^2\hbar^4[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)] - ik^2\hbar^4(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)}{[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)]^2 + (m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)^2} \times \frac{k^2\hbar^4[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)] + ik^2\hbar^4(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)}{[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)]^2 + (m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)^2} \\
&= \frac{\{k^2\hbar^4[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)]\}^2 + \left\{k^2\hbar^4(m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)\right\}^2}{\left\{[k^2\hbar^4 - m^2\alpha^2(1-\cos 4ak)]^2 + (m^2\alpha^2\sin 4ak - 2km\alpha\hbar^2)^2\right\}^2} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 2k^2m^2\alpha^2\hbar^4 + 2m^4\alpha^4 - 2m^2\alpha^2\left[(m^2\alpha^2 - k^2\hbar^4)\cos 4ak + 2km\alpha\hbar^2\sin 4ak\right]}
\end{aligned}$$

Therefore, replacing k with $\sqrt{2mE}/\hbar$, the transmission coefficient is

$$\boxed{T = \frac{2\hbar^4E^2}{2\hbar^4E^2 + 2\hbar^2m\alpha^2E + m^2\alpha^4 + m\alpha^2\left[(2\hbar^2E - m\alpha^2)\cos\frac{4a\sqrt{2mE}}{\hbar} - 2\hbar\alpha\sqrt{2mE}\sin\frac{4a\sqrt{2mE}}{\hbar}\right]}}.$$

In order to illustrate the function's behavior, plot T versus $\hbar^2 E$ for the special case that $a = \hbar$, $m = \hbar^2$, and $\alpha = 1/\hbar$.



Now return to the final set of equations involving C_1, C_2, and C_5.

$$\begin{cases} \dfrac{2m\alpha}{\hbar^2}\left(C_1 e^{-ika} + C_2 e^{ika}\right) = 2iC_1 k e^{-ika} - 2i\dfrac{C_5 e^{ika} - C_1 e^{-3ika} - C_2 e^{-ika}}{e^{ika} - e^{-3ika}} k e^{-ika} \\ \left(\dfrac{2m\alpha}{\hbar^2} + ik\right) C_5 e^{ika} = -iC_1 k e^{-3ika} - iC_2 k e^{-ika} + i\dfrac{C_5 e^{ika} - C_1 e^{-3ika} - C_2 e^{-ika}}{e^{ika} - e^{-3ika}} k(e^{ika} + e^{-3ika}) \end{cases}$$

This time solve the first equation for C_5 rather than C_2

$$C_5 = \frac{im\alpha}{k\hbar^2}(C_1 + C_2 e^{2iak})(1 - e^{-4iak}) + (C_1 + C_2 e^{-2iak})$$

and then plug it into the second equation.

$$\begin{aligned} &\left(\frac{2m\alpha}{\hbar^2} + ik\right)\left[\frac{im\alpha}{k\hbar^2}(C_1 + C_2 e^{2iak})(1 - e^{-4iak}) + (C_1 + C_2 e^{-2iak})\right] e^{ika} \\ &\qquad = -iC_1 k e^{-3ika} - iC_2 k e^{-ika} \\ &\qquad\qquad + i\frac{\left[\frac{im\alpha}{k\hbar^2}(C_1 + C_2 e^{2iak})(1 - e^{-4iak}) + (C_1 + C_2 e^{-2iak})\right] e^{ika} - C_1 e^{-3ika} - C_2 e^{-ika}}{e^{ika} - e^{-3ika}} k(e^{ika} + e^{-3ika}) \end{aligned}$$

Solve for C_2.

$$C_2 = m\alpha \frac{ik\hbar^2(1 + e^{4iak}) + m\alpha(1 - e^{4iak})}{m^2\alpha^2 e^{6iak} - (m\alpha + ik\hbar^2)^2 e^{2iak}} C_1$$

Divide both sides by C_1.

$$\frac{C_2}{C_1} = m\alpha\frac{ik\hbar^2(1+e^{4iak})+m\alpha(1-e^{4iak})}{m^2\alpha^2e^{6iak}-(m\alpha+ik\hbar^2)^2e^{2iak}}$$

$$= m\alpha\frac{ik\hbar^2(1+\cos 4ak+i\sin 4ak)+m\alpha(1-\cos 4ak-i\sin 4ak)}{m^2\alpha^2(\cos 6ak+i\sin 6ak)-(m^2\alpha^2+2ikm\alpha\hbar^2-k^2\hbar^4)(\cos 2ak+i\sin 2ak)}$$

$$= m\alpha\frac{[m\alpha(1-\cos 4ak)-k\hbar^2\sin 4ak]+i[k\hbar^2(1+\cos 4ak)-m\alpha\sin 4ak]}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]-i[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]}$$

$$= m\alpha\frac{[m\alpha(1-\cos 4ak)-k\hbar^2\sin 4ak]+i[k\hbar^2(1+\cos 4ak)-m\alpha\sin 4ak]}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]-i[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]}$$
$$\times\frac{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]+i[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]+i[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]}$$

$$= m\alpha\frac{m\alpha\{m\alpha[m\alpha\cos 6ak+k\hbar^2(5\sin 2ak+\sin 6ak)]-(m^2\alpha^2+4k^2\hbar^4)\cos 2ak\}+2i[k^2\hbar^4-m^2\alpha^2(1-\cos 4ak)](k\hbar^2\cos 2ak-m\alpha\sin 2ak)}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]^2+[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]^2}$$

Multiply both sides by the complex conjugate to get the reflection coefficient.

$$\left|\frac{C_2}{C_1}\right|^2 = m\alpha\frac{m\alpha\{m\alpha[m\alpha\cos 6ak+k\hbar^2(5\sin 2ak+\sin 6ak)]-(m^2\alpha^2+4k^2\hbar^4)\cos 2ak\}+2i[k^2\hbar^4-m^2\alpha^2(1-\cos 4ak)](k\hbar^2\cos 2ak-m\alpha\sin 2ak)}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]^2+[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]^2}$$
$$\times m\alpha\frac{m\alpha\{m\alpha[m\alpha\cos 6ak+k\hbar^2(5\sin 2ak+\sin 6ak)]-(m^2\alpha^2+4k^2\hbar^4)\cos 2ak\}-2i[k^2\hbar^4-m^2\alpha^2(1-\cos 4ak)](k\hbar^2\cos 2ak-m\alpha\sin 2ak)}{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]^2+[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]^2}$$

$$= m^2\alpha^2\frac{m^2\alpha^2\{m\alpha[m\alpha\cos 6ak+k\hbar^2(5\sin 2ak+\sin 6ak)]-(m^2\alpha^2+4k^2\hbar^4)\cos 2ak\}^2+4[k^2\hbar^4-m^2\alpha^2(1-\cos 4ak)]^2(k\hbar^2\cos 2ak-m\alpha\sin 2ak)^2}{\{[m\alpha(m\alpha\cos 6ak+2k\hbar^2\sin 2ak)+(k^2\hbar^4-m^2\alpha^2)\cos 2ak]^2+[2km\alpha\hbar^2\cos 2ak-m^2\alpha^2(\sin 6ak-\sin 2ak)-k^2\hbar^4\sin 2ak]^2\}^2}$$

$$= \frac{4(k\hbar^2\cos 2ak-m\alpha\sin 2ak)^2}{k^4\hbar^8+2k^2m^2\alpha^2\hbar^4+2m^4\alpha^4-2m^2\alpha^2\left[(m^2\alpha^2-k^2\hbar^4)\cos 4ak+2km\alpha\hbar^2\sin 4ak\right]}$$

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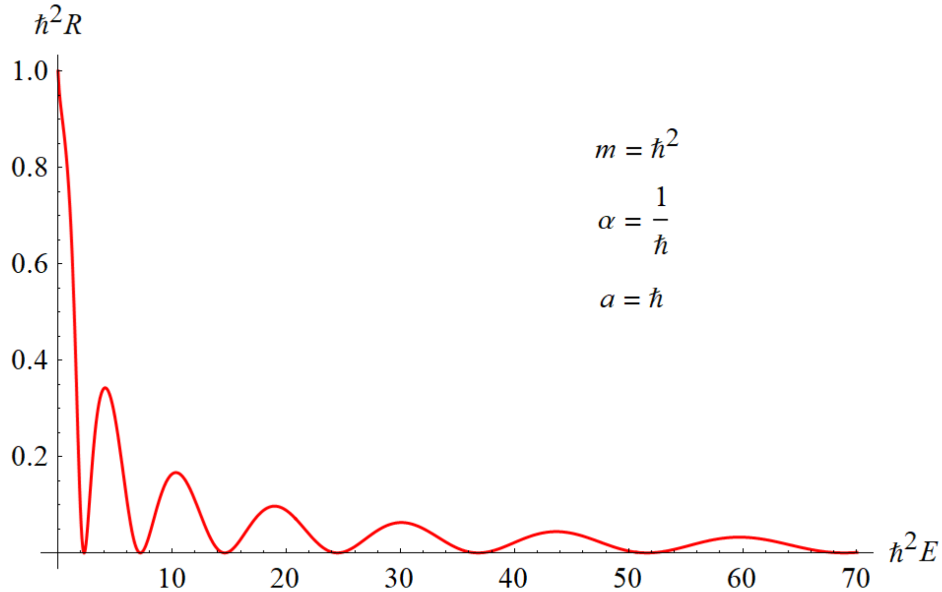
Therefore, replacing k with $\sqrt{2mE}/\hbar$, the reflection coefficient is

$$R = \frac{\frac{2}{m^2}\left(\hbar\sqrt{2mE}\cos\frac{2a\sqrt{2mE}}{\hbar} - m\alpha\sin\frac{2a\sqrt{2mE}}{\hbar}\right)^2}{2\hbar^4E^2 + 2\hbar^2m\alpha^2E + m^2\alpha^4 + m\alpha^2\left[(2\hbar^2E - m\alpha^2)\cos\frac{4a\sqrt{2mE}}{\hbar} - 2\hbar\alpha\sqrt{2mE}\sin\frac{4a\sqrt{2mE}}{\hbar}\right]}.$$

In order to illustrate the function's behavior, plot $\hbar^2R$ versus $\hbar^2E$ for the special case that $a = \hbar$, $m = \hbar^2$, and $\alpha = 1/\hbar$.



Note that $R + T \neq 1$ for this double-well potential unless $\hbar = 1$, $m = 1$, $a = 1$, and $\alpha = 1$.