Problem 2.31

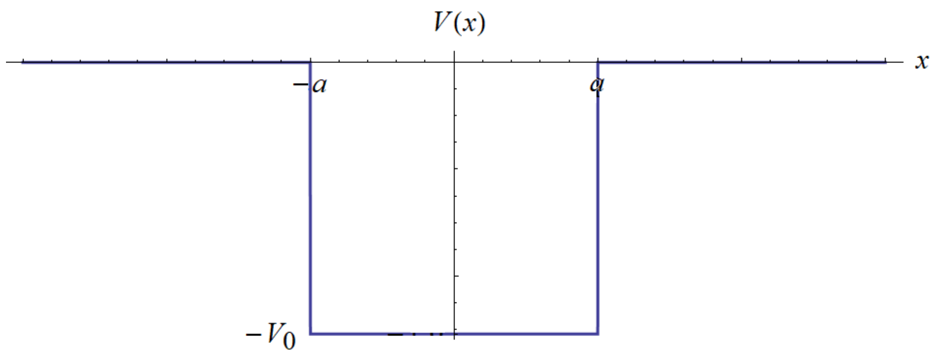
The Dirac delta function can be thought of as the limiting case of a rectangle of area 1, as the height goes to infinity and the width goes to zero. Show that the delta-function well (Equation 2.117) is a "weak" potential (even though it is infinitely deep), in the sense that $z_0 \to 0$. Determine the bound state energy for the delta-function potential, by treating it as the limit of a finite square well. Check that your answer is consistent with Equation 2.132. Also show that Equation 2.172 reduces to Equation 2.144 in the appropriate limit.

Solution

The finite square well potential is

$$\begin{cases} 0 & \text{if } x < -a \\ -V_0 & \text{if } -a \leq x \leq a \\ 0 & \text{if } x > a \end{cases}$$

and is plotted below versus x.



To turn it into the delta-function well, $V(x) = -\alpha\delta(x)$, set

$$V_0 = \frac{\alpha}{\epsilon} \quad \text{and} \quad a = \frac{\epsilon}{2}$$

and then take the limit as $\epsilon \to 0$. a is $\epsilon/2$ rather than ϵ so that the area of the rectangle is α. Consequently,

$$z_0 = \frac{\sqrt{2mV_0}}{\hbar}a = \frac{\sqrt{2m}}{\hbar}\sqrt{\frac{\alpha}{\epsilon}}\left(\frac{\epsilon}{2}\right) = \frac{1}{\hbar}\sqrt{\frac{m\alpha}{2}\epsilon}.$$

Upon taking the limit as $\epsilon \to 0$, we obtain

$$\lim_{\epsilon \to 0} z_0 = 0,$$

which means the delta-function well is a weak potential. It was found in Problem 2.29 that there are no odd bound states if $z_0 < \pi/2$. There are only even bound states, then, and their energies are found from Equation 2.159 (page 72) in the textbook.

$$\tan z = \sqrt{(z_0/z)^2 - 1} \tag{2.159}$$

Substitute

$$z_0 = \frac{\sqrt{2mV_0}}{\hbar}a \quad \text{and} \quad z = \frac{\sqrt{2m(E + V_0)}}{\hbar}a$$

into this formula.

$$\tan\left[\frac{\sqrt{2m(E+V_0)}}{\hbar}a\right] = \sqrt{\frac{\frac{2mV_0}{\hbar^2}a^2}{\frac{2m(E+V_0)}{\hbar^2}a^2} - 1}$$

$$= \sqrt{\frac{V_0}{E+V_0} - 1}$$

Now set

$$V_0 = \frac{\alpha}{\epsilon} \quad \text{and} \quad a = \frac{\epsilon}{2}.$$

As a result,

$$\tan\left[\frac{\sqrt{2m\left(E+\frac{\alpha}{\epsilon}\right)}}{\hbar}\frac{\epsilon}{2}\right] = \sqrt{\frac{\frac{\alpha}{\epsilon}}{E+\frac{\alpha}{\epsilon}} - 1}$$

$$\tan\left(\frac{1}{\hbar}\sqrt{\frac{m\epsilon^2 E + m\epsilon\alpha}{2}}\right) = \sqrt{\frac{1}{\frac{\epsilon E}{\alpha}+1} - 1}.$$

Since ϵ is a very small positive number, $m\epsilon^2 E$ is negligible compared to $m\epsilon\alpha$.

$$\tan\left(\frac{1}{\hbar}\sqrt{\frac{m\epsilon\alpha}{2}}\right) = \sqrt{\frac{1}{\frac{\epsilon E}{\alpha}+1} - 1}$$

In addition, the fact that the tangent's argument is very small allows us to linearize the function. Recall that the Taylor series expansion of tangent about the origin is

$$\tan X = X + \frac{X^3}{3} + \frac{2X^5}{15} + \cdots.$$

For very small X, the higher-order terms on the right side are negligible: $\tan X \approx X$.

$$\frac{1}{\hbar}\sqrt{\frac{m\epsilon\alpha}{2}} = \sqrt{\frac{1}{\frac{\epsilon E}{\alpha}+1} - 1}$$

Solve for E.

$$E = -\frac{m\alpha^2}{m\epsilon\alpha + 2\hbar^2}$$

Take the limit as $\epsilon \to 0$.

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

This is the energy of the bound state for $V(x) = -\alpha\delta(x)$, which is consistent with Equation 2.132 on page 66.

Equation 2.172 (page 73) gives the transmission coefficient for the finite square well potential.

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right) \tag{2.172}$$

Set

$$V_0 = \frac{\alpha}{\epsilon} \quad \text{and} \quad a = \frac{\epsilon}{2}.$$

Consequently,

$$\begin{aligned} T^{-1} &= 1 + \frac{\frac{\alpha^2}{\epsilon^2}}{4E\left(E+\frac{\alpha}{\epsilon}\right)} \sin^2\left[\frac{2\left(\frac{\epsilon}{2}\right)}{\hbar}\sqrt{2m\left(E+\frac{\alpha}{\epsilon}\right)}\right] \\ &= 1 + \frac{\alpha^2}{4\epsilon^2 E^2 + 4\epsilon E\alpha} \sin^2\left(\frac{1}{\hbar}\sqrt{2m\epsilon^2 E + 2m\epsilon\alpha}\right). \end{aligned}$$

Since ϵ is a very small positive number, $4\epsilon^2 E^2$ is negligible compared to $4\epsilon E\alpha$; also, $2m\epsilon^2 E$ is negligible compared to $2m\epsilon\alpha$.

$$T^{-1} = 1 + \frac{\alpha^2}{4\epsilon E\alpha} \sin^2\left(\frac{1}{\hbar}\sqrt{2m\epsilon\alpha}\right)$$

In addition, the fact that the sine's argument is very small allows us to linearize the function. Recall that the Taylor series expansion of sine about the origin is

$$\sin X = X - \frac{X^3}{6} + \frac{X^5}{120} - \cdots.$$

For very small X, the higher-order terms on the right side are negligible: $\sin X \approx X$.

$$\begin{aligned} T^{-1} &= 1 + \frac{\alpha^2}{4\epsilon E\alpha}\left(\frac{1}{\hbar}\sqrt{2m\epsilon\alpha}\right)^2 \\ &= 1 + \frac{\alpha}{4\epsilon E}\left(\frac{2m\epsilon\alpha}{\hbar^2}\right) \\ &= 1 + \frac{m\alpha^2}{2\hbar^2 E} \end{aligned}$$

Inverting both sides yields

$$T = \frac{1}{1 + (m\alpha^2/2\hbar^2 E)},$$

which is consistent with Equation 2.144 (page 68).