Problem 2.33

Determine the transmission coefficient for a rectangular *barrier* (same as Equation 2.148, only with $V(x) = +V_0 > 0$ in the region $-a < x < a$). Treat separately the three cases $E < V_0$, $E = V_0$, and $E > V_0$ (note that the wave function inside the barrier is different in the three cases). *Partial answer:* for $E < V_0$,[52]

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right).$$

Solution

The governing equation for the wave function $\Psi(x,t)$ is the Schrödinger equation.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

For a rectangular barrier,

$$V(x,t) = V(x) = \begin{cases} 0 & \text{if } x < -a \\ V_0 & \text{if } -a \leq x \leq a \\ 0 & \text{if } x > a \end{cases},$$

which means the PDE becomes

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi(x,t).$$

Since information about the eigenstates and their corresponding energies is desired, the method of separation of variables is opted for. This method works because Schrödinger's equation and its associated boundary conditions (Ψ and its derivatives tend to zero as $x \to \pm\infty$) are linear and homogeneous. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] + V(x)[\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) + V(x)\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ in order to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x)$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) = E$$

[52]This is a good example of tunneling—*classically* the particle would bounce back.

As a result of using the method of separation of variables, the Schrödinger equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) &= E \end{aligned}\right\}$$

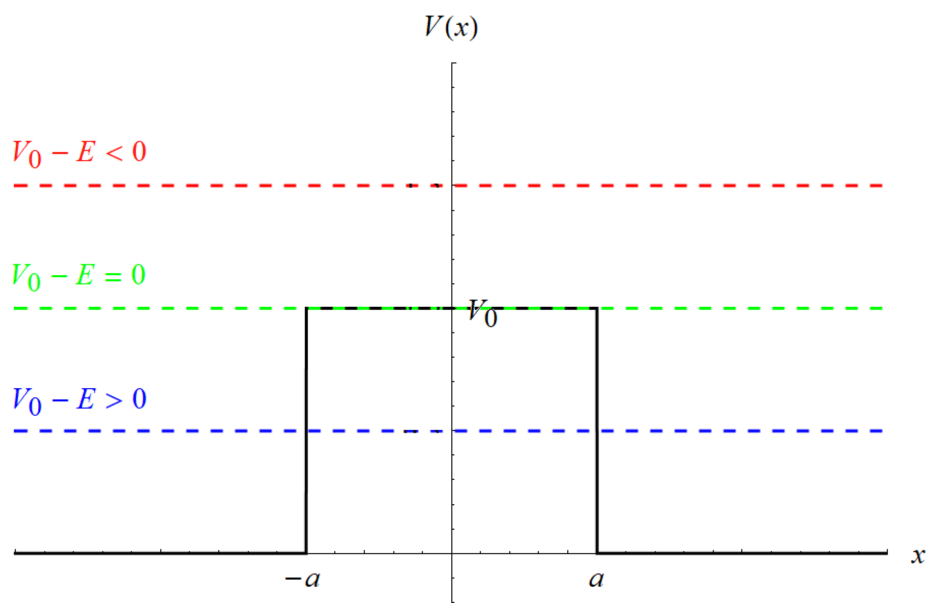
Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi.$$

Split up the ODE over the intervals that $V(x)$ is defined on.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-E)\psi, \quad |x| > a \qquad\qquad \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\psi, \quad -a \le x \le a$$

The solution for ψ on the interval $-a \le x \le a$ depends on whether $V_0 - E > 0$, $V_0 - E = 0$, or $V_0 - E < 0$. Each of these cases will be examined in turn.



Note that scattering states correspond to $E > 0$. In each case, the aim is to determine the reflection and transmission coefficients.

Case I: $V_0 - E > 0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi, \quad |x| > a \qquad\qquad \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\psi, \quad -a \le x \le a$$

In this case, the general solution on $-a \le x \le a$ can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < -a \\ C\cosh \ell x + D\sinh \ell x & \text{if } -a \le x \le a \\ Fe^{ikx} + Ge^{-ikx} & \text{if } x > a \end{cases}$$

Here

$$k = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad \ell = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

Solving the ODE in t yields $\phi(t) = e^{-iEt/\hbar}$, which means the product solution is a linear combination of waves travelling to the left and to the right (on $x < -a$ and $x > a$).

$$\psi(x)\phi(t) = \begin{cases} Ae^{i(kx - Et/\hbar)} + Be^{-i(kx + Et/\hbar)} & \text{if } x < -a \\ Ce^{-iEt/\hbar}\cosh \ell x + De^{-iEt/\hbar}\sinh \ell x & \text{if } -a \le x \le a \\ Fe^{i(kx - Et/\hbar)} + Ge^{-i(kx + Et/\hbar)} & \text{if } x > a \end{cases}$$

Assuming there's a plane wave incident from the left, $G = 0$, and the reflection and transmission coefficients are $R = |B/A|^2$ and $T = |F/A|^2$, respectively. Require the wave function [and consequently $\psi(x)$] to be continuous at $x = -a$ and $x = a$ to determine two of the constants.

$$\lim_{x \to -a^-} \psi(x) = \lim_{x \to -a^+} \psi(x) : \quad Ae^{-ika} + Be^{ika} = C\cosh \ell a - D\sinh \ell a$$
$$\lim_{x \to +a^-} \psi(x) = \lim_{x \to +a^+} \psi(x) : \quad C\cosh \ell a + D\sinh \ell a = Fe^{ika} + Ge^{-ika}$$

Integrate both sides of the TISE with respect to x from $-a - \epsilon$ to $-a + \epsilon$, where ϵ is a really small positive number, to determine one more constant.

$$\begin{aligned} \int_{-a-\epsilon}^{-a+\epsilon} \frac{d^2\psi}{dx^2}\,dx &= \int_{-a-\epsilon}^{-a+\epsilon} \frac{2m}{\hbar^2}[V(x) - E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon} &= \int_{-a-\epsilon}^{-a} \frac{2m}{\hbar^2}(-E)\psi(x)\,dx + \int_{-a}^{-a+\epsilon} \frac{2m}{\hbar^2}(V_0 - E)\psi(x)\,dx \\ &= -\frac{2mE}{\hbar^2}\psi(-a)\int_{-a-\epsilon}^{-a} dx + \frac{2m}{\hbar^2}(V_0 - E)\psi(-a)\int_{-a}^{-a+\epsilon} dx \\ &= -\frac{2mE}{\hbar^2}\psi(-a)\epsilon + \frac{2m}{\hbar^2}(V_0 - E)\psi(-a)\epsilon \end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{-a^-}^{-a^+} = 0$$

It turns out that $\partial\Psi/\partial x$ is continuous at $x = -a$ as well.

$$\lim_{x \to -a^-} \frac{d\psi}{dx} = \lim_{x \to -a^+} \frac{d\psi}{dx} : \quad ik(Ae^{-ika} - Be^{ika}) = \ell(-C\sinh \ell a + D\cosh \ell a)$$

Integrate both sides of the TISE with respect to x from $a-\epsilon$ to $a+\epsilon$ to determine one more constant.

$$\begin{aligned}\int_{a-\epsilon}^{a+\epsilon} \frac{d^2\psi}{dx^2}\,dx &= \int_{a-\epsilon}^{a+\epsilon} \frac{2m}{\hbar^2}[V(x)-E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon} &= \int_{a-\epsilon}^{a} \frac{2m}{\hbar^2}(V_0-E)\psi(x)\,dx + \int_a^{a+\epsilon} \frac{2m}{\hbar^2}(-E)\psi(x)\,dx \\ &= \frac{2m}{\hbar^2}(V_0-E)\psi(a)\int_{a-\epsilon}^{a} dx - \frac{2mE}{\hbar^2}\psi(a)\int_a^{a+\epsilon} dx \\ &= \frac{2m(V_0-E)}{\hbar^2}\psi(a)\epsilon - \frac{2mE}{\hbar^2}\psi(a)\epsilon\end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{a^-}^{a^+} = 0$$

$\partial\Psi/\partial x$ also happens to be continuous at $x = a$.

$$\lim_{x\to a^-}\frac{d\psi}{dx} = \lim_{x\to a^+}\frac{d\psi}{dx}: \quad \ell(C\sinh \ell a + D\cosh \ell a) = ik(Fe^{ika} - Ge^{-ika})$$

To summarize, there are four equations involving A, B, C, D, and F. G is set equal to zero.

$$\begin{cases} Ae^{-ika} + Be^{ika} = C\cosh \ell a - D\sinh \ell a \\ C\cosh \ell a + D\sinh \ell a = Fe^{ika} \\ ik(Ae^{-ika} - Be^{ika}) = \ell(-C\sinh \ell a + D\cosh \ell a) \\ \ell(C\sinh \ell a + D\cosh \ell a) = ik(Fe^{ika}) \end{cases}$$

Solve the second and fourth equations for C and D. To get C, multiply both sides of the second by $\cosh \ell a$ and multiply both sides of the fourth by $(\sinh \ell a)/\ell$.

$$\begin{cases} C\cosh^2 \ell a + D\sinh \ell a\cosh \ell a = Fe^{ika}\cosh \ell a \\ C\sinh^2 \ell a + D\sinh \ell a\cosh \ell a = \dfrac{ik}{\ell}Fe^{ika}\sinh \ell a \end{cases}$$

Subtract the respective sides to eliminate D.

$$C(\cosh^2 \ell a - \sinh^2 \ell a) = Fe^{ika}\left(\cosh \ell a - \frac{ik}{\ell}\sinh \ell a\right)$$

Use the fact that $\cosh^2 \ell a - \sinh^2 \ell a = 1$.

$$C = Fe^{ika}\left(\cosh \ell a - \frac{ik}{\ell}\sinh \ell a\right)$$

To get D, multiply both sides of the second by $\sinh \ell a$ and multiply both sides of the fourth by $(\cosh \ell a)/\ell$.

$$\begin{cases} C\sinh \ell a\cosh \ell a + D\sinh^2 \ell a = Fe^{ika}\sinh \ell a \\ C\sinh \ell a\cosh \ell a + D\cosh^2 \ell a = \dfrac{ik}{\ell}Fe^{ika}\cosh \ell a \end{cases}$$

Subtract the respective sides to eliminate C.

$$D(\cosh^2 \ell a - \sinh^2 \ell a) = Fe^{ika}\left(\frac{ik}{\ell}\cosh \ell a - \sinh \ell a\right)$$

$$D = Fe^{ika}\left(\frac{ik}{\ell}\cosh \ell a - \sinh \ell a\right)$$

Substitute these formulas for C and D into the first equation.

$$\begin{aligned} Ae^{-ika} + Be^{ika} &= C\cosh \ell a - D\sinh \ell a \\ &= Fe^{ika}\left(\cosh \ell a - \frac{ik}{\ell}\sinh \ell a\right)\cosh \ell a - Fe^{ika}\left(\frac{ik}{\ell}\cosh \ell a - \sinh \ell a\right)\sinh \ell a \\ &= Fe^{ika}\left(\cosh^2 \ell a - \frac{2ik}{\ell}\sinh \ell a \cosh \ell a + \sinh^2 \ell a\right) \\ &= Fe^{ika}\left(\cosh 2\ell a - \frac{ik}{\ell}\sinh 2\ell a\right) \end{aligned}$$

Substitute these formulas for C and D into the third equation.

$$\begin{aligned} \frac{ik}{\ell}(Ae^{-ika} - Be^{ika}) &= -C\sinh \ell a + D\cosh \ell a \\ &= -Fe^{ika}\left(\cosh \ell a - \frac{ik}{\ell}\sinh \ell a\right)\sinh \ell a + Fe^{ika}\left(\frac{ik}{\ell}\cosh \ell a - \sinh \ell a\right)\cosh \ell a \\ &= Fe^{ika}\left[-2\sinh \ell a\cosh \ell a + \frac{ik}{\ell}(\sinh^2 \ell a + \cosh^2 \ell a)\right] \\ &= Fe^{ika}\left(-\sinh 2\ell a + \frac{ik}{\ell}\cosh 2\ell a\right) \end{aligned}$$

Consequently, the first and third equations become

$$\begin{cases} Ae^{-ika} + Be^{ika} = Fe^{ika}\left(\cosh 2\ell a - \dfrac{ik}{\ell}\sinh 2\ell a\right) \\ Ae^{-ika} - Be^{ika} = Fe^{ika}\left(\cosh 2\ell a - \dfrac{\ell}{ik}\sinh 2\ell a\right) \end{cases}.$$

Subtract the respective sides to eliminate A.

$$\begin{aligned} 2Be^{ika} &= Fe^{ika}\left(-\frac{ik}{\ell}\sinh 2\ell a + \frac{\ell}{ik}\sinh 2\ell a\right) \\ &= Fe^{ika}\left(\frac{-i^2k^2 + \ell^2}{ik\ell}\right)\sinh 2\ell a \\ &= Fe^{ika}\left(\frac{k^2 + \ell^2}{ik\ell}\right)\sinh 2\ell a \end{aligned}$$

Solve for B.

$$B = F\left(\frac{k^2 + \ell^2}{2ik\ell}\right)\sinh 2\ell a$$

Add the respective sides to eliminate B.

$$\begin{aligned}2Ae^{-ika} &= Fe^{ika}\left[2\cosh 2\ell a - \left(\frac{ik}{\ell} + \frac{\ell}{ik}\right)\sinh 2\ell a\right] \\ &= Fe^{ika}\left(2\cosh 2\ell a - \frac{i^2k^2 + \ell^2}{ik\ell}\sinh 2\ell a\right) \\ &= Fe^{ika}\left(2\cosh 2\ell a + \frac{k^2 - \ell^2}{ik\ell}\sinh 2\ell a\right) \\ &= Fe^{ika}\left(2\cosh 2\ell a - i\frac{k^2 - \ell^2}{k\ell}\sinh 2\ell a\right)\end{aligned}$$

Solve for F.

$$F = \frac{e^{-2ika}A}{\cosh 2\ell a - i\frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}$$

The transmission coefficient can now be determined.

$$\begin{aligned}T = \left|\frac{F}{A}\right|^2 = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^* &= \left[\frac{e^{-2ika}}{\cosh 2\ell a - i\frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}\right]\left[\frac{e^{2ika}}{\cosh 2\ell a + i\frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}\right] \\ &= \frac{1}{\cosh^2 2\ell a + \frac{(k^2-\ell^2)^2}{4k^2\ell^2}\sinh^2 2\ell a}\end{aligned}$$

Invert both sides and then plug in the formulas for k and ℓ.

$$\begin{aligned}T^{-1} &= \cosh^2 2\ell a + \frac{(k^2-\ell^2)^2}{4k^2\ell^2}\sinh^2 2\ell a \\ &= \cosh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} + \frac{(V_0-2E)^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} \\ &= \cosh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} + \frac{V_0^2 - 4EV_0 + 4E^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} \\ &= \cosh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} + \frac{V_0^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} - \frac{4EV_0 - 4E^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar}\end{aligned}$$

Therefore, the transmission coefficient for the case that $V_0 - E > 0$ is

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar}.}$$

Combine the formulas for B and F.

$$B = F\left(\frac{k^2+\ell^2}{2ik\ell}\right)\sinh 2\ell a = \frac{e^{-2ika}A}{\cosh 2\ell a - i\frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}\left(\frac{k^2+\ell^2}{2ik\ell}\right)\sinh 2\ell a = \frac{e^{-2ika}A\left(\frac{k^2+\ell^2}{2k\ell}\right)\sinh 2\ell a}{i\cosh 2\ell a + \frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}$$

Now the reflection coefficient can be determined.

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^* = \left[\frac{e^{-2ika}\left(\frac{k^2+\ell^2}{2k\ell}\right)\sinh 2\ell a}{i\cosh 2\ell a + \frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}\right]\left[\frac{e^{2ika}\left(\frac{k^2+\ell^2}{2k\ell}\right)\sinh 2\ell a}{-i\cosh 2\ell a + \frac{k^2-\ell^2}{2k\ell}\sinh 2\ell a}\right]$$

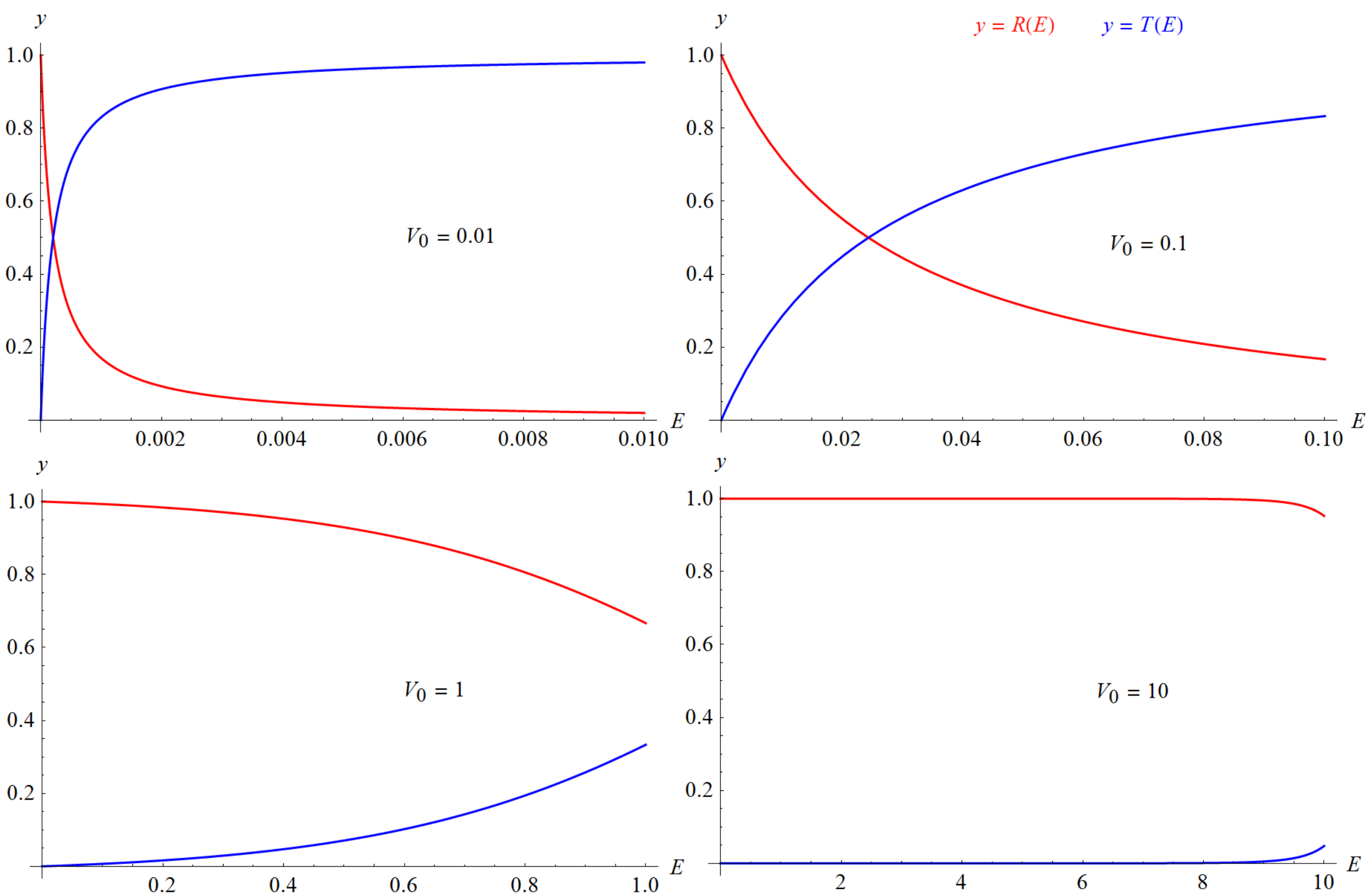
$$= \frac{\frac{(k^2+\ell^2)^2}{4k^2\ell^2}\sinh^2 2\ell a}{\cosh^2 2\ell a + \frac{(k^2-\ell^2)^2}{4k^2\ell^2}\sinh^2 2\ell a} = \frac{\frac{(k^2+\ell^2)^2}{4k^2\ell^2}\sinh^2 2\ell a}{T^{-1}}$$

$$= \frac{\frac{V_0^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar}}{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar}} = \frac{1}{\frac{4E(V_0-E)}{V_0^2}\operatorname{csch}^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar} + 1}$$

Therefore, the reflection coefficient for the case that $V_0 - E > 0$ is

$$\boxed{R^{-1} = 1 + \frac{4E(V_0-E)}{V_0^2}\operatorname{csch}^2 \frac{2a\sqrt{2m(V_0-E)}}{\hbar}.}$$

Note that $R + T = 1$. Below are plots of $y = T(E)$ (in blue) and $y = R(E)$ (in red) versus E for $m = \hbar$, $a = \sqrt{\hbar}$, and various magnitudes of V_0.



Case II: $V_0 - E = 0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mV_0}{\hbar^2}\psi, \quad |x| > a \qquad\qquad \frac{d^2\psi}{dx^2} = 0, \quad -a \leq x \leq a$$

In this case, the general solution on $-a \leq x \leq a$ is a straight line.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < -a \\ Cx + D & \text{if } -a \leq x \leq a \\ Fe^{ikx} + Ge^{-ikx} & \text{if } x > a \end{cases}$$

Here

$$k = \frac{\sqrt{2mV_0}}{\hbar}.$$

Solving the ODE in t yields $\phi(t) = e^{-iEt/\hbar}$, which means the product solution is a linear combination of waves travelling to the left and to the right (on $x < -a$ and $x > a$).

$$\psi(x)\phi(t) = \begin{cases} Ae^{i(kx-Et/\hbar)} + Be^{-i(kx+Et/\hbar)} & \text{if } x < -a \\ Ce^{-iEt/\hbar}x + De^{-iEt/\hbar} & \text{if } -a \leq x \leq a \\ Fe^{i(kx-Et/\hbar)} + Ge^{-i(kx+Et/\hbar)} & \text{if } x > a \end{cases}$$

Assuming there's a plane wave incident from the left, $G = 0$, and the reflection and transmission coefficients are $R = |B/A|^2$ and $T = |F/A|^2$, respectively. Require the wave function [and consequently $\psi(x)$] to be continuous at $x = -a$ and $x = a$ to determine two of the constants.

$$\lim_{x\to -a^-} \psi(x) = \lim_{x\to -a^+} \psi(x) : \quad Ae^{-ika} + Be^{ika} = -Ca + D$$

$$\lim_{x\to +a^-} \psi(x) = \lim_{x\to +a^+} \psi(x) : \quad Ca + D = Fe^{ika} + Ge^{-ika}$$

Integrate both sides of the TISE with respect to x from $-a-\epsilon$ to $-a+\epsilon$, where ϵ is a really small positive number, to determine one more constant.

$$\begin{aligned} \int_{-a-\epsilon}^{-a+\epsilon} \frac{d^2\psi}{dx^2}\,dx &= \int_{-a-\epsilon}^{-a+\epsilon} \frac{2m}{\hbar^2}[V(x) - E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{-a-\epsilon}^{-a+\epsilon} &= \int_{-a-\epsilon}^{-a} \frac{2m}{\hbar^2}(-E)\psi(x)\,dx + \int_{-a}^{-a+\epsilon} \frac{2m}{\hbar^2}(V_0 - E)\psi(x)\,dx \\ &= -\frac{2mV_0}{\hbar^2}\psi(-a)\int_{-a-\epsilon}^{-a} dx \\ &= -\frac{2mV_0}{\hbar^2}\psi(-a)\epsilon \end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{-a^-}^{-a^+} = 0$$

It turns out that $\partial\Psi/\partial x$ is continuous at $x = -a$ as well.

$$\lim_{x\to -a^-} \frac{d\psi}{dx} = \lim_{x\to -a^+} \frac{d\psi}{dx} : \quad ik(Ae^{-ika} - Be^{ika}) = C$$

Integrate both sides of the TISE with respect to x from $a-\epsilon$ to $a+\epsilon$ to determine one more constant.

$$\begin{aligned}
\int_{a-\epsilon}^{a+\epsilon} \frac{d^2\psi}{dx^2}\,dx &= \int_{a-\epsilon}^{a+\epsilon} \frac{2m}{\hbar^2}[V(x)-E]\psi(x)\,dx \\
\left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon} &= \int_{a-\epsilon}^{a} \frac{2m}{\hbar^2}(V_0-E)\psi(x)\,dx + \int_{a}^{a+\epsilon} \frac{2m}{\hbar^2}(-E)\psi(x)\,dx \\
&= -\frac{2mV_0}{\hbar^2}\psi(a)\int_a^{a+\epsilon} dx \\
&= -\frac{2mV_0}{\hbar^2}\psi(a)\epsilon
\end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{a^-}^{a^+} = 0$$

$\partial\Psi/\partial x$ also happens to be continuous at $x = a$.

$$\lim_{x\to a^-}\frac{d\psi}{dx} = \lim_{x\to a^+}\frac{d\psi}{dx}: \quad C = ik(Fe^{ika} - Ge^{-ika})$$

To summarize, there are four equations involving A, B, C, D, and F. G is set equal to zero.

$$\begin{cases}
Ae^{-ika} + Be^{ika} = -Ca + D \\
Ca + D = Fe^{ika} \\
ik(Ae^{-ika} - Be^{ika}) = C \\
C = ik(Fe^{ika})
\end{cases}$$

Solve the second and fourth equations for C and D.

$$\begin{aligned}
C &= ikFe^{ika} \\
D &= Fe^{ika}(1-ika)
\end{aligned}$$

Substitute these results into the first and third equations.

$$\begin{cases}
Ae^{-ika} + Be^{ika} = Fe^{ika}(1-2ika) \\
Ae^{-ika} - Be^{ika} = Fe^{ika}
\end{cases}$$

Subtract the respective sides to get B.

$$2Be^{ika} = Fe^{ika}(-2ika)$$

$$B = -ikaF$$

Add the respective sides to get A.

$$2Ae^{-ika} = Fe^{ika}(2-2ika)$$

$$F = \frac{e^{-2ika}A}{1-ika}$$

The transmission coefficient can now be found.

$$T = \left|\frac{F}{A}\right|^2 = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^* = \left(\frac{e^{-2ika}}{1-ika}\right)\left(\frac{e^{2ika}}{1+ika}\right) = \frac{1}{1+k^2a^2} = \frac{1}{1+\frac{2mV_0a^2}{\hbar^2}}$$

Therefore, the transmission coefficient for the case that $V_0 - E = 0$ is

$$\boxed{T = \frac{\hbar^2}{\hbar^2 + 2ma^2V_0}.}$$

Now combine the formulas for B and F.

$$B = -ikaF = -ika\frac{e^{-2ika}A}{1-ika}$$

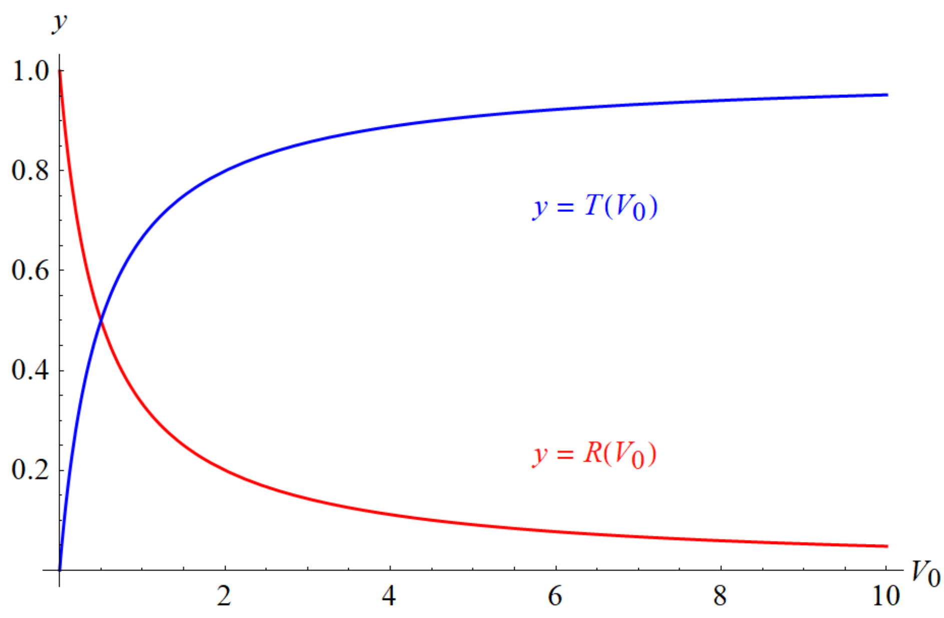
The reflection coefficient is

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^* = \left(-ika\frac{e^{-2ika}}{1-ika}\right)\left(ika\frac{e^{2ika}}{1+ika}\right) = \frac{k^2a^2}{1+k^2a^2} = \frac{\frac{2mV_0a^2}{\hbar^2}}{1+\frac{2mV_0a^2}{\hbar^2}}.$$

Therefore, the reflection coefficient for the case that $V_0 - E = 0$ is

$$\boxed{R = \frac{2ma^2V_0}{\hbar^2 + 2ma^2V_0}.}$$

Note that $R + T = 1$. Below are plots of $y = T(V_0)$ (in blue) and $y = R(V_0)$ (in red) versus V_0 for $m = \hbar$ and $a = \sqrt{\hbar}$.



Case III: $V_0 - E < 0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi, \quad |x| > a \qquad\qquad \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi, \quad -a \le x \le a$$

In this case, the general solution on $-a \le x \le a$ can be written in terms of sine and cosine.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < -a \\ C_0 \sin lx + D_0 \cos lx & \text{if } -a \le x \le a \\ Fe^{ikx} + Ge^{-ikx} & \text{if } x > a \end{cases}$$

Here

$$k = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

This scattering state was analyzed in Problem 2.32, so the same formulas for R and T in terms of k and l can be used. Plug in these new formulas for k and l.

$$\begin{aligned} T^{-1} &= \cos^2 2la + \frac{(k^2 + l^2)^2}{4k^2l^2} \sin^2 2la \\ &= \cos^2 2la + \frac{(2E - V_0)^2}{4E(E - V_0)} \sin^2 \frac{2a\sqrt{2m(E - V_0)}}{\hbar} \\ &= \cos^2 2la + \frac{4E^2 - 4EV_0 + V_0^2}{4E(E - V_0)} \sin^2 \frac{2a\sqrt{2m(E - V_0)}}{\hbar} \\ &= \cos^2 2la + \frac{4E^2 - 4EV_0}{4E(E - V_0)} \sin^2 2la + \frac{V_0^2}{4E(E - V_0)} \sin^2 \frac{2a\sqrt{2m(E - V_0)}}{\hbar} \end{aligned}$$

Therefore, the transmission coefficient for the case that $V_0 - E < 0$ is

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E - V_0)} \sin^2 \frac{2a\sqrt{2m(E - V_0)}}{\hbar}.}$$

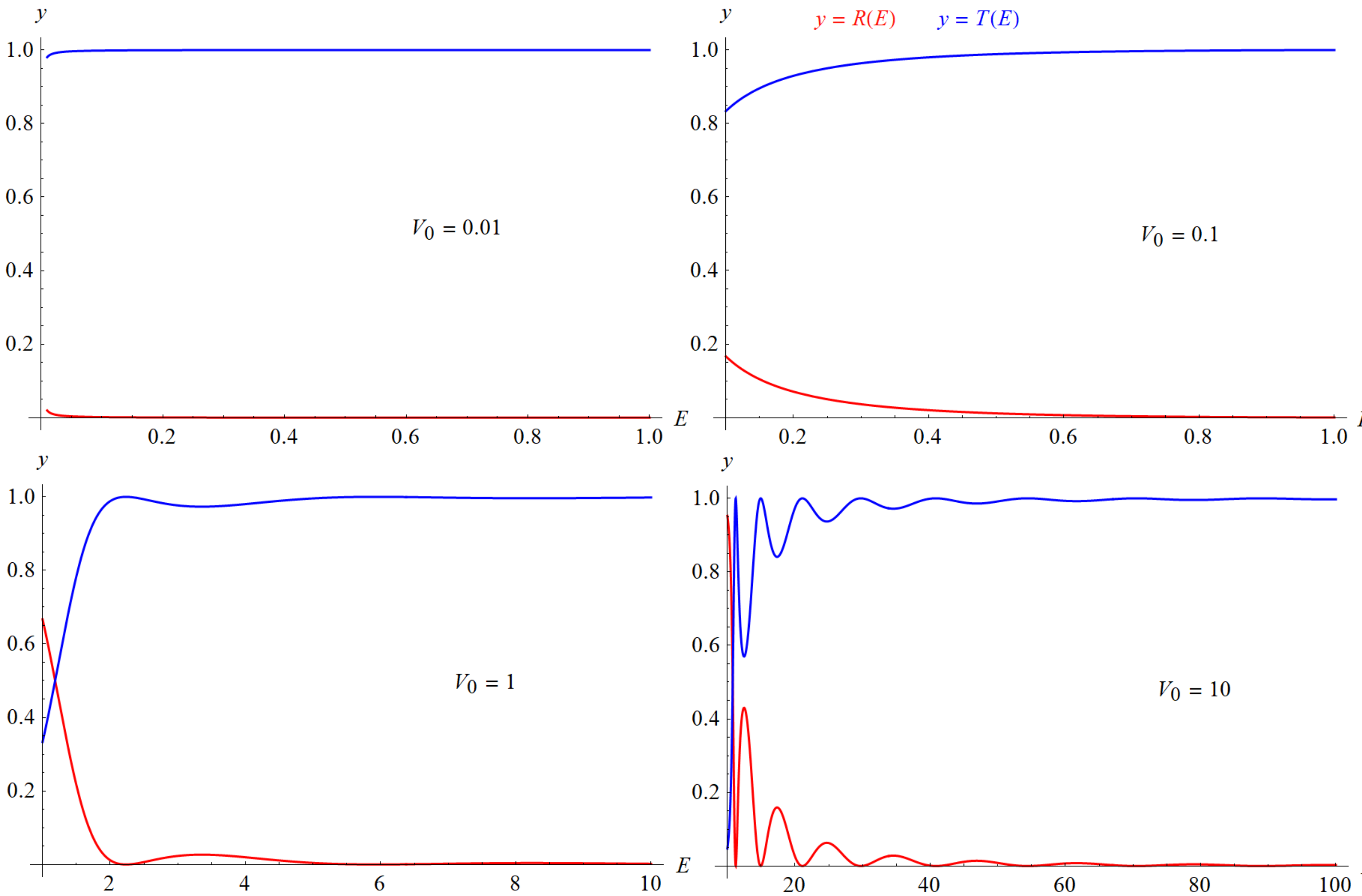
$$\begin{aligned} R &= \frac{(l^2 - k^2)^2}{4k^2l^2} \frac{\sin^2 2la}{T^{-1}} \\ &= \frac{\frac{V_0^2}{4E(E-V_0)} \sin^2 \frac{2a\sqrt{2m(E-V_0)}}{\hbar}}{1 + \frac{V_0^2}{4E(E-V_0)} \sin^2 \frac{2a\sqrt{2m(E-V_0)}}{\hbar}} \\ &= \frac{1}{\frac{4E(E-V_0)}{V_0^2} \csc^2 \frac{2a\sqrt{2m(E-V_0)}}{\hbar} + 1} \end{aligned}$$

Therefore, the reflection coefficient for the case that $V_0 - E < 0$ is

$$\boxed{R^{-1} = 1 + \frac{4E(E - V_0)}{V_0^2} \csc^2 \frac{2a\sqrt{2m(E - V_0)}}{\hbar}.}$$

Note that $R + T = 1$.

Below are plots of $y = T(E)$ (in blue) and $y = R(E)$ (in red) versus E for $m = \hbar$, $a = \sqrt{\hbar}$, and various magnitudes of V_0.



In conclusion, for a rectangular barrier,

$$R^{-1} = \begin{cases} 1 + \dfrac{4E(V_0 - E)}{V_0^2} \operatorname{csch}^2 \dfrac{2a\sqrt{2m(V_0 - E)}}{\hbar} & \text{if } E < V_0 \\ \dfrac{\hbar^2 + 2ma^2V_0}{2ma^2V_0} & \text{if } E = V_0 \\ 1 + \dfrac{4E(E - V_0)}{V_0^2} \csc^2 \dfrac{2a\sqrt{2m(E - V_0)}}{\hbar} & \text{if } E > V_0 \end{cases}$$

$$T^{-1} = \begin{cases} 1 + \dfrac{V_0^2}{4E(V_0 - E)} \sinh^2 \dfrac{2a\sqrt{2m(V_0 - E)}}{\hbar} & \text{if } E < V_0 \\ \dfrac{\hbar^2 + 2ma^2V_0}{\hbar^2} & \text{if } E = V_0 \\ 1 + \dfrac{V_0^2}{4E(E - V_0)} \sin^2 \dfrac{2a\sqrt{2m(E - V_0)}}{\hbar} & \text{if } E > V_0 \end{cases}.$$