Problem 2.38

(a) Show that the wave function of a particle in the infinite square well returns to its original form after a quantum **revival time** $T = 4ma^2/\pi\hbar$. That is: $\Psi(x,T) = \Psi(x,0)$ for any state (*not* just a stationary state).

(b) What is the *classical* revival time, for a particle of energy E bouncing back and forth between the walls?

(c) For what energy are the two revival times equal?[55]

Solution

The general solution to the Schrödinger equation for a particle in the infinite square well is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \exp\left(-i\frac{n^2\pi^2\hbar}{2ma^2}t\right) \sin\frac{n\pi x}{a}.$$

$\Psi(x,0)$ is the wave function's original form. T, the quantum revival time, is defined such that

$$\Psi(x,0) = \Psi(x,T): \quad \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \exp\left(-i\frac{n^2\pi^2\hbar}{2ma^2}T\right) \sin\frac{n\pi x}{a}.$$

This implies that

$$\exp\left(-i\frac{n^2\pi^2\hbar}{2ma^2}T\right) = 1.$$

Use Euler's formula to write this in terms of sine and cosine.

$$\cos\left(\frac{n^2\pi^2\hbar}{2ma^2}T\right) - i\sin\left(\frac{n^2\pi^2\hbar}{2ma^2}T\right) = 1$$

Match the real and imaginary components of both sides.

$$\left.\begin{aligned} \cos\left(\frac{n^2\pi^2\hbar}{2ma^2}T\right) &= 1 \\ \sin\left(\frac{n^2\pi^2\hbar}{2ma^2}T\right) &= 0 \end{aligned}\right\}$$

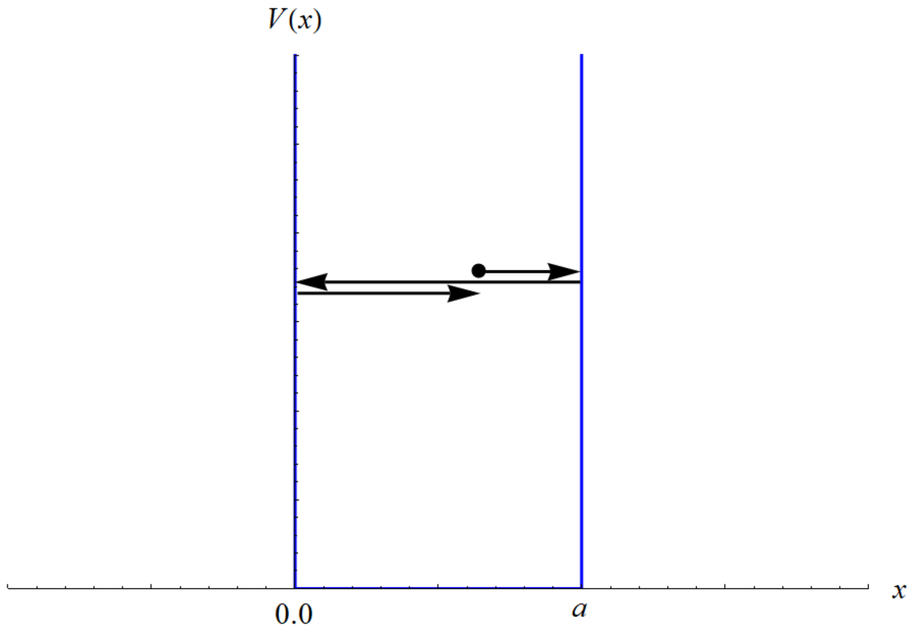
As n^2 is an integer, both equations are satisfied if

$$\frac{\pi^2\hbar}{2ma^2}T = 2\pi.$$

The right side should technically be $2\pi q$, where q is any integer, but because we want the smallest revival time after $t = 0$, we set $q = 1$. Therefore,

$$T = \frac{4ma^2}{\pi\hbar}.$$

[55]The fact that the classical and quantum revival times bear no obvious relation to one another (and the quantum one doesn't even depend on the energy) is a curious paradox; see D. F. Styer, *Am. J. Phys.* **69**, 56 (2001).



Classically, a particle bouncing back and forth between the walls of an infinite square well goes a distance $2a$ before it reaches its initial state again.

$$2a = vT$$

Solve for T.

$$T = \frac{2a}{v}$$

Within the well

$$E = PE + KE = 0 + \frac{1}{2}mv^2 = \frac{1}{2}mv^2 \quad \rightarrow \quad v = \pm\sqrt{\frac{2E}{m}}.$$

Choose the positive sign for v and substitute the formula into the one for T.

$$T = \frac{2a}{\sqrt{\frac{2E}{m}}}$$

Therefore, the classical revival time is

$$T = a\sqrt{\frac{2m}{E}}.$$

For the quantum and classical revival times to be equal, the energy must be

$$\frac{4ma^2}{\pi\hbar} = a\sqrt{\frac{2m}{E}}$$

$$\sqrt{E} = a\sqrt{2m}\frac{\pi\hbar}{4ma^2}$$

$$E = a^2(2m)\frac{\pi^2\hbar^2}{16m^2a^4}$$

$$E = \frac{\pi^2\hbar^2}{8ma^2}.$$