Problem 2.43

Solve the time-independent Schrödinger equation for a centered infinite square well with a delta-function barrier in the middle:

$$V(x) = \begin{cases} \alpha\delta(x), & -a < x < +a, \\ \infty, & |x| \geq a. \end{cases}$$

Treat the even and odd wave functions separately. Don't bother to normalize them. Find the allowed energies (graphically, if necessary). How do they compare with the corresponding energies in the absence of the delta function? Explain why the odd solutions are not affected by the delta function. Comment on the limiting cases $\alpha \to 0$ and $\alpha \to \infty$.

Solution

Schrödinger's equation governs the time evolution of the wave function $\Psi(x,t)$.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

Split up the PDE over the intervals that $V(x,t) = V(x)$ is defined on.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + (\infty)\Psi(x,t), \quad |x| \geq a \qquad i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \alpha\delta(x)\Psi(x,t), \quad -a < x < a$$

Only $\Psi(x,t) = 0$ satisfies the one for $|x| \geq a$. Because the wave function must be continuous, $\Psi(-a,t) = 0$ and $\Psi(a,t) = 0$ are boundary conditions for the remaining PDE on $-a < x < a$.

$$\begin{aligned} & i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \alpha\delta(x)\Psi(x,t), \quad -a < x < a \\ & \Psi(-a,t) = 0 \\ & \Psi(a,t) = 0 \end{aligned}$$

Since information about the eigenstates and their corresponding energies is desired, the method of separation of variables is opted for. This method works because Schrödinger's equation and its associated boundary conditions are linear and homogeneous. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] + \alpha\delta(x)[\psi(x)\phi(t)]$$

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) + \alpha\delta(x)\psi(x)\phi(t)$$

and the boundary conditions.

$$\begin{aligned} \Psi(-a,t) = 0 \quad &\to \quad \psi(-a)\phi(t) = 0 \quad \to \quad \psi(-a) = 0 \\ \Psi(a,t) = 0 \quad &\to \quad \psi(a)\phi(t) = 0 \quad \to \quad \psi(a) = 0 \end{aligned}$$

Divide both sides of the PDE by $\psi(x)\phi(t)$ in order to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \alpha\delta(x)$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \alpha\delta(x) = E$$

As a result of using the method of separation of variables, the Schrödinger equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \alpha\delta(x) &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi, \quad -a < x < a.$$

For all values of $x \neq 0$, the delta function vanishes.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(-E)\psi, \quad x \neq 0$$

Check to see if zero is an eigenvalue: $E = 0$.

$$\frac{d^2\psi}{dx^2} = 0, \quad x \neq 0$$

The general solution is

$$\psi(x) = \begin{cases} C_1x + C_2 & \text{if } -a < x < 0 \\ C_3x + C_4 & \text{if } 0 < x < a \end{cases}.$$

Apply the boundary conditions to determine two constants.

$$\psi(-a) = -C_1a + C_2 = 0 \tag{1}$$

$$\psi(a) = C_3a + C_4 = 0 \tag{2}$$

Require the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ to determine one more.

$$\lim_{x\to 0^-}\psi(x) = \lim_{x\to 0^+}\psi(x) : \quad C_2 = C_4 \tag{3}$$

Integrate both sides of the TISE with respect to x from $-\epsilon$ to ϵ, where ϵ is a really small positive number, to determine one more.

$$\begin{aligned} \int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx &= \int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} &= \frac{2m}{\hbar^2}\alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx \\ C_3 - C_1 &= \frac{2m}{\hbar^2}\alpha\psi(0) \\ &= \frac{2m}{\hbar^2}\alpha C_2 \end{aligned} \tag{4}$$

Solving equations (1), (2), (3), and (4) yields $C_1 = 0$, $C_2 = 0$, $C_3 = 0$, and $C_4 = 0$, which results in the trivial solution, $\psi(x) = 0$. Therefore, zero is not an eigenvalue. Now check to see if there are negative eigenvalues: $E = -\gamma^2$.

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\gamma^2\psi, \quad x \neq 0$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\psi(x) = \begin{cases} C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } -a < x < 0 \\ C_7 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_8 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } 0 < x < a \end{cases}$$

Because the given function for $V(x)$ is even, $\psi(x)$ is either even or odd [see part (c) of Problem 2.1]. Consider the case where $\psi(x)$ is even first:

$$\psi(x) = \psi(-x)$$

$$C_7 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_8 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} = C_5 \cosh \frac{\sqrt{2m}\gamma(-x)}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma(-x)}{\hbar}$$

$$C_7 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_8 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} = C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar}.$$

Match the coefficients to determine two constants.

$$C_7 = C_5$$
$$C_8 = -C_6$$

So the even eigenstate is

$$\psi(x) = \begin{cases} C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } -a < x < 0 \\ C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } 0 < x < a \end{cases}.$$

Apply the boundary conditions to determine one constant.

$$\psi(-a) = C_5 \cosh \frac{\sqrt{2m}\gamma a}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma a}{\hbar} = 0 \tag{5}$$

$$\psi(a) = C_5 \cosh \frac{\sqrt{2m}\gamma a}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma a}{\hbar} = 0 \tag{6}$$

Requiring the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ determines nothing.

$$\lim_{x\to 0^-} \psi(x) = \lim_{x\to 0^+} \psi(x): \quad C_5 = C_5 \tag{7}$$

Integrate both sides of the TISE with respect to x from $-\epsilon$ to ϵ to determine one more.

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2}\, dx &= \int_{-\epsilon}^{\epsilon} \frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi(x)\, dx \\ \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} &= \frac{2m}{\hbar^2}\left[\alpha \int_{-\epsilon}^{\epsilon} \delta(x)\psi(x)\, dx + \gamma^2 \int_{-\epsilon}^{\epsilon} \psi(x)\, dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \gamma^2\psi(0)\int_{-\epsilon}^{\epsilon} dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \gamma^2\psi(0)(2\epsilon)\right] \end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} = \frac{2m}{\hbar^2}\alpha\psi(0) \quad \rightarrow \quad -\frac{2C_6\sqrt{2m}\gamma}{\hbar} = \frac{2m}{\hbar^2}\alpha C_5 \tag{8}$$

Solve equation (8) for C_6.

$$C_6 = -C_5\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\gamma}$$

Substitute this into equation (5).

$$C_5\cosh\frac{\sqrt{2m}\gamma a}{\hbar} + C_5\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\gamma}\sinh\frac{\sqrt{2m}\gamma a}{\hbar} = 0$$

To avoid the trivial solution, insist that $C_5 \neq 0$.

$$\tanh\frac{\sqrt{2m}\gamma a}{\hbar} = -\sqrt{\frac{2}{m}}\frac{\hbar\gamma}{\alpha}$$

Set

$$z = \frac{\sqrt{2m}\gamma a}{\hbar} \quad \rightarrow \quad \frac{\hbar^2 z}{\alpha a\sqrt{2m}} = \frac{\hbar\gamma}{\alpha}.$$

As a result,

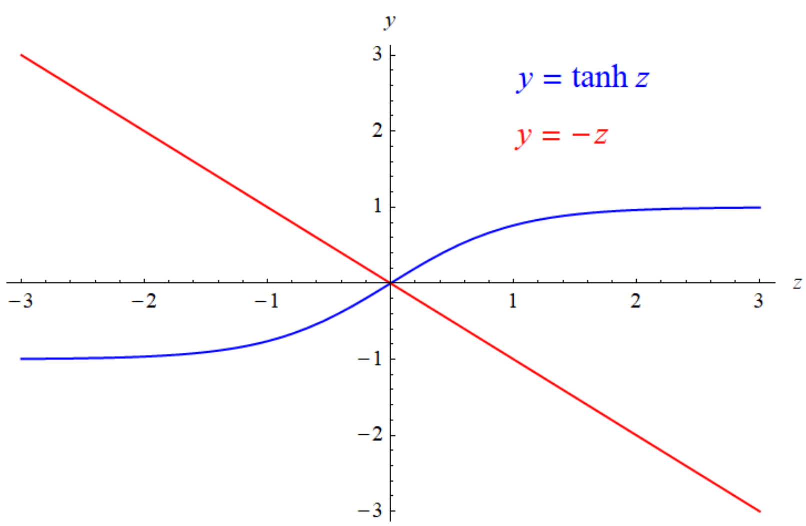
$$\tanh z = -\sqrt{\frac{2}{m}}\left(\frac{\hbar^2 z}{\alpha a\sqrt{2m}}\right) = -\frac{\hbar^2 z}{\alpha a m}.$$

In addition, set

$$z_0 = \frac{\alpha a m}{\hbar^2}$$

so that the equation for the eigenvalues becomes

$$\tanh z = -\frac{z}{z_0}.$$



Since there are no nonzero values of z for which the graphs intersect, regardless of what z_0 is, there are no negative eigenvalues if $\psi(x)$ is even.

Now consider the case where $\psi(x)$ is odd:

$$\psi(x) = -\psi(-x)$$

$$C_7 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_8 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} = -C_5 \cosh \frac{\sqrt{2m}\gamma(-x)}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma(-x)}{\hbar}$$

$$C_7 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_8 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} = -C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar}.$$

Match the coefficients to determine two constants.

$$C_7 = -C_5$$
$$C_8 = C_6$$

So the odd eigenstate is

$$\psi(x) = \begin{cases} C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } -a < x < 0 \\ -C_5 \cosh \frac{\sqrt{2m}\gamma x}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma x}{\hbar} & \text{if } 0 < x < a \end{cases}.$$

Apply the boundary conditions to determine one constant.

$$\psi(-a) = C_5 \cosh \frac{\sqrt{2m}\gamma a}{\hbar} - C_6 \sinh \frac{\sqrt{2m}\gamma a}{\hbar} = 0 \tag{9}$$

$$\psi(a) = -C_5 \cosh \frac{\sqrt{2m}\gamma a}{\hbar} + C_6 \sinh \frac{\sqrt{2m}\gamma a}{\hbar} = 0 \tag{10}$$

Require the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ to determine one constant.

$$\lim_{x \to 0^-} \psi(x) = \lim_{x \to 0^+} \psi(x) : \quad C_5 = -C_5 \quad \Rightarrow \quad C_5 = 0 \tag{11}$$

Integrating both sides of the TISE with respect to x from $-\epsilon$ to ϵ determines nothing new.

$$\int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2}\,dx = \int_{-\epsilon}^{\epsilon} \frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi(x)\,dx$$

$$\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} = \frac{2m}{\hbar^2}\left[\alpha \int_{-\epsilon}^{\epsilon} \delta(x)\psi(x)\,dx + \gamma^2 \int_{-\epsilon}^{\epsilon} \psi(x)\,dx\right]$$

$$= \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \gamma^2\psi(0)\int_{-\epsilon}^{\epsilon} dx\right]$$

$$= \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \gamma^2\psi(0)(2\epsilon)\right]$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} = \frac{2m}{\hbar^2}\alpha\psi(0) \quad \to \quad 0 = \frac{2m}{\hbar^2}\alpha C_5 \quad \to \quad C_5 = 0 \tag{12}$$

Equation (9) becomes

$$-C_6 \sinh \frac{\sqrt{2m}\gamma a}{\hbar} = 0.$$

There are no nonzero values of γ that make the hyperbolic sine vanish, so $C_6 = 0$. This results in the trivial solution, $\psi(x) = 0$, so there are no negative eigenvalues if $\psi(x)$ is odd.

Now check to see if there are positive eigenvalues: $E = \mu^2$.

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\mu^2\psi, \quad x \neq 0$$

The general solution can be written in terms of sine and cosine.

$$\psi(x) = \begin{cases} C_9 \cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ C_{11}\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{12}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}$$

Because the given function for $V(x)$ is even, $\psi(x)$ is either even or odd [see part (c) of Problem 2.1]. Consider the case where $\psi(x)$ is even first:

$$\psi(x) = \psi(-x)$$

$$C_{11}\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{12}\sin\frac{\sqrt{2m}\mu x}{\hbar} = C_9\cos\frac{\sqrt{2m}\mu(-x)}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu(-x)}{\hbar}$$

$$C_{11}\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{12}\sin\frac{\sqrt{2m}\mu x}{\hbar} = C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar}.$$

Match the coefficients to determine two constants.

$$C_{11} = C_9$$
$$C_{12} = -C_{10}$$

So the even eigenstate is

$$\psi(x) = \begin{cases} C_9 \cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}.$$

Apply the boundary conditions to determine one constant.

$$\psi(-a) = C_9\cos\frac{\sqrt{2m}\mu a}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0 \tag{13}$$

$$\psi(a) = C_9\cos\frac{\sqrt{2m}\mu a}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0 \tag{14}$$

Requiring the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ determines nothing.

$$\lim_{x\to 0^-}\psi(x) = \lim_{x\to 0^+}\psi(x): \quad C_9 = C_9 \tag{15}$$

Integrate both sides of the TISE with respect to x from $-\epsilon$ to ϵ to determine one more.

$$\begin{aligned}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx &= \int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} &= \frac{2m}{\hbar^2}\left[\alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx - \mu^2\int_{-\epsilon}^{\epsilon}\psi(x)\,dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) - \mu^2\psi(0)\int_{-\epsilon}^{\epsilon}dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) - \mu^2\psi(0)(2\epsilon)\right]\end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} = \frac{2m}{\hbar^2}\alpha\psi(0) \quad \to \quad -\frac{2C_{10}\sqrt{2m}\mu}{\hbar} = \frac{2m}{\hbar^2}\alpha C_9 \tag{16}$$

Solve equation (16) for C_{10}.

$$C_{10} = -C_9\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\mu}$$

Substitute this into equation (13).

$$C_9\cos\frac{\sqrt{2m}\mu a}{\hbar} + C_9\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\mu}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0$$

To avoid the trivial solution, insist that $C_9 \neq 0$.

$$\tan\frac{\sqrt{2m}\mu a}{\hbar} = -\sqrt{\frac{2}{m}}\frac{\hbar\mu}{\alpha}$$

Set

$$z = \frac{\sqrt{2m}\mu a}{\hbar} \quad \to \quad \frac{\hbar^2 z}{\alpha a\sqrt{2m}} = \frac{\hbar\mu}{\alpha}.$$

As a result,

$$\tan z = -\sqrt{\frac{2}{m}}\left(\frac{\hbar^2 z}{\alpha a\sqrt{2m}}\right) = -\frac{\hbar^2 z}{\alpha a m}.$$

Finally, set

$$z_0 = \frac{\alpha a m}{\hbar^2}$$

so that the equation for the eigenvalues becomes

$$\tan z = -\frac{z}{z_0}.$$

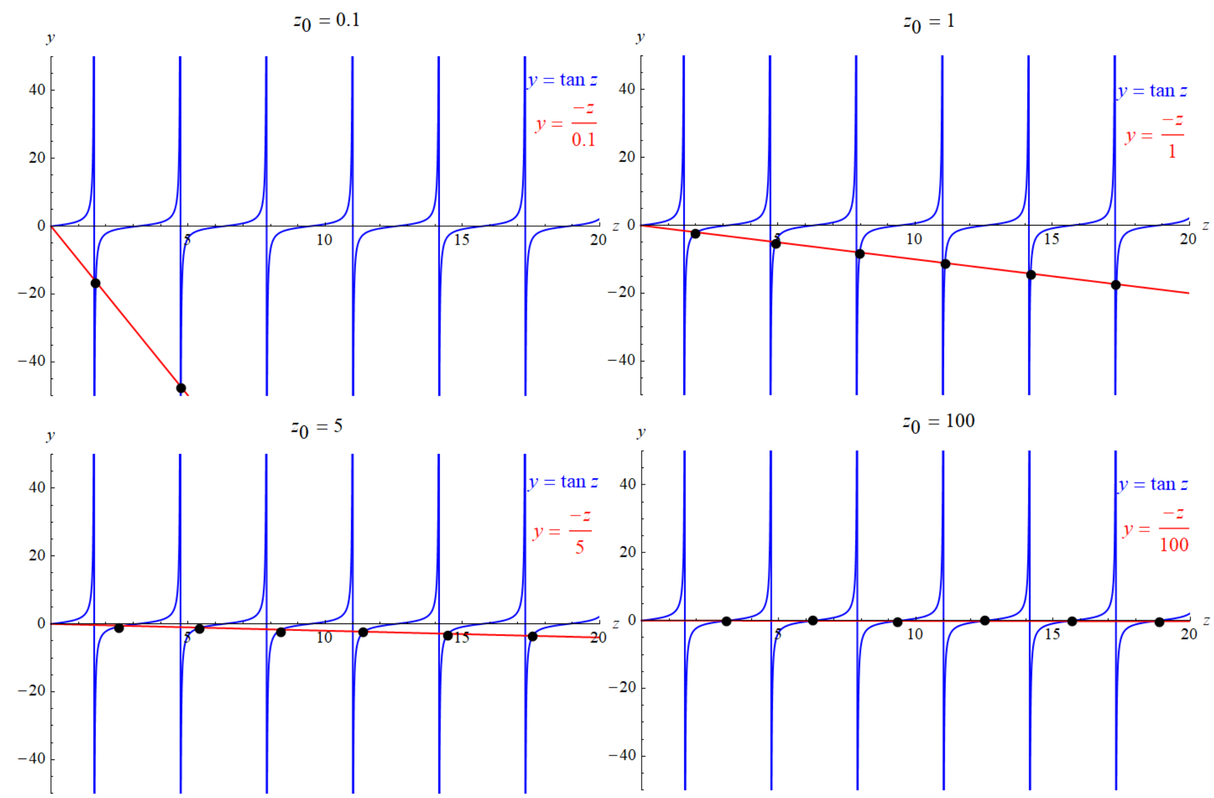
There are intersections at nonzero values of z as shown on the next page, so there are even eigenstates.

$$\psi(x) = \begin{cases} C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}$$

$$= \begin{cases} C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} - C_9\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\mu}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_9\sqrt{\frac{m}{2}}\frac{\alpha}{\hbar\mu}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}$$

C_9 is arbitrary and is chosen so that the integral of $[\psi(x)]^2$ over $-a < x < a$ is 1.

$$1 = \int_{-a}^{a}[\psi(x)]^2\,dx \quad \Rightarrow \quad C_9 = \frac{2\sqrt{2\mu^3}}{\sqrt{4\mu\left(\alpha + 2a\mu^2 + \frac{ma\alpha^2}{\hbar^2}\right) - 4\alpha\mu\cos\frac{2a\mu\sqrt{2m}}{\hbar} + \sqrt{\frac{2}{m}}\frac{2\mu^2\hbar^2 - m\alpha^2}{\hbar}\sin\frac{2a\mu\sqrt{2m}}{\hbar}}}$$



Based on these graphs, the intersections occur between odd integer multiples of $\pi/2$ and integer multiples of π, depending on what z_0 is numerically.

$$\frac{(2n-1)\pi}{2} < z < n\pi, \quad n = 1, 2, \ldots$$

Substitute the formula for z and solve for μ.

$$\frac{(2n-1)\pi}{2} < \frac{\sqrt{2m}\mu a}{\hbar} < n\pi$$

$$\frac{(2n-1)\pi\hbar}{2a\sqrt{2m}} < \mu < \frac{2n\pi\hbar}{2a\sqrt{2m}}$$

Therefore, since $E = \mu^2$, the eigenvalues are

$$\frac{(2n-1)^2\pi^2\hbar^2}{2m(2a)^2} < E_n < \frac{(2n)^2\pi^2\hbar^2}{2m(2a)^2}, \quad n = 1, 2, \ldots.$$

These lower and upper bounds for E_n are the limiting values for $\alpha \to 0$ and $\alpha \to \infty$, respectively.

$$\lim_{\alpha \to 0} E_n = \frac{(2n-1)^2\pi^2\hbar^2}{2m(2a)^2}$$

$$\lim_{\alpha \to \infty} E_n = \frac{(2n)^2\pi^2\hbar^2}{2m(2a)^2} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

In the limit as $\alpha \to 0$, these are the odd energies for a centered infinite square well (Problem 2.36). This second limit indicates that a centered delta function of infinite strength effectively cuts the well in half.

Now consider the case where $\psi(x)$ is odd:

$$\psi(x) = -\psi(-x)$$

$$C_{11}\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{12}\sin\frac{\sqrt{2m}\mu x}{\hbar} = -C_9\cos\frac{\sqrt{2m}\mu(-x)}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu(-x)}{\hbar}$$

$$C_{11}\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{12}\sin\frac{\sqrt{2m}\mu x}{\hbar} = -C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar}.$$

Match the coefficients to determine two constants.

$$C_{11} = -C_9$$
$$C_{12} = C_{10}$$

So the odd eigenstate is

$$\psi(x) = \begin{cases} C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ -C_9\cos\frac{\sqrt{2m}\mu x}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}.$$

Apply the boundary conditions to determine one constant.

$$\psi(-a) = C_9\cos\frac{\sqrt{2m}\mu a}{\hbar} - C_{10}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0 \tag{17}$$

$$\psi(a) = -C_9\cos\frac{\sqrt{2m}\mu a}{\hbar} + C_{10}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0 \tag{18}$$

Require the wave function [and consequently $\psi(x)$] to be continuous at $x = 0$ to determine one constant.

$$\lim_{x\to 0^-}\psi(x) = \lim_{x\to 0^+}\psi(x): \quad C_9 = -C_9 \quad \Rightarrow \quad C_9 = 0 \tag{19}$$

Integrating both sides of the TISE with respect to x from $-\epsilon$ to ϵ determines nothing new.

$$\begin{aligned}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx &= \int_{-\epsilon}^{\epsilon}\frac{2m}{\hbar^2}[\alpha\delta(x) - E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} &= \frac{2m}{\hbar^2}\left[\alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx - \mu^2\int_{-\epsilon}^{\epsilon}\psi(x)\,dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) - \mu^2\psi(0)\int_{-\epsilon}^{\epsilon}dx\right] \\ &= \frac{2m}{\hbar^2}\left[\alpha\psi(0) - \mu^2\psi(0)(2\epsilon)\right]\end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{0^-}^{0^+} = \frac{2m}{\hbar^2}\alpha\psi(0) \quad \to \quad 0 = \frac{2m}{\hbar^2}\alpha C_9 \quad \to \quad C_9 = 0 \tag{20}$$

Equation (17) becomes

$$-C_{10}\sin\frac{\sqrt{2m}\mu a}{\hbar} = 0.$$

To avoid the trivial solution, insist that $C_{10} \neq 0$.

$$\sin \frac{\sqrt{2m}\mu a}{\hbar} = 0$$

$$\frac{\sqrt{2m}\mu a}{\hbar} = n\pi, \quad n = 1, 2, \dots$$

$$\mu = \frac{n\pi\hbar}{a\sqrt{2m}}$$

Therefore, since $E = \mu^2$, the eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{(2n)^2\pi^2\hbar^2}{2m(2a)^2}, \quad n = 1, 2, \dots.$$

These are the even energies for a centered infinite square well (Problem 2.36). The odd eigenstates associated with them are

$$\psi(x) = \begin{cases} C_9 \cos \frac{\sqrt{2m}\mu x}{\hbar} + C_{10} \sin \frac{\sqrt{2m}\mu x}{\hbar} & \text{if } -a < x < 0 \\ -C_9 \cos \frac{\sqrt{2m}\mu x}{\hbar} + C_{10} \sin \frac{\sqrt{2m}\mu x}{\hbar} & \text{if } 0 < x < a \end{cases}$$

$$= C_{10} \sin \frac{\sqrt{2m}\mu x}{\hbar}.$$

C_{10} is arbitrary and is chosen so that the integral of $[\psi(x)]^2$ over $-a < x < a$ is 1.

$$1 = \int_{-a}^{a} [\psi(x)]^2 \, dx \quad \Rightarrow \quad C_{10} = \frac{2\sqrt{\mu}}{\sqrt{4a\mu - \sqrt{\frac{2}{m}}\hbar \sin \frac{2a\mu\sqrt{2m}}{\hbar}}} = \frac{1}{\sqrt{a}}$$

This makes the odd eigenstates

$$\psi(x) = \frac{1}{\sqrt{a}} \sin \frac{n\pi x}{a}.$$

The odd solutions are not affected by the delta function because it only acts at $x = 0$, and odd functions are zero at $x = 0$ by definition.