

Problem 2.5

A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)].$$

- Normalize $\Psi(x, 0)$. (That is, find A . This is very easy, if you exploit the orthonormality of ψ_1 and ψ_2 . Recall that, having normalized Ψ at $t = 0$, you can rest assured that it *stays* normalized—if you doubt this, check it explicitly after doing part (b).)
- Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \pi^2 \hbar / 2ma^2$.
- Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $a/2$, go directly to jail.)
- Compute $\langle p \rangle$. (As Peter Lorre would say, “Do it ze *kveek* vay, Johnny!”)
- If you measured the energy of the particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H . How does it compare with E_1 and E_2 ?

Solution

In Problem 2.3 the general solution to the Schrödinger equation for the infinite square well potential,

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases},$$

was found to be

$$\Psi(x, t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \exp\left(-i \frac{\hbar \pi^2 n^2}{2ma^2} t\right) \sin \frac{n\pi x}{a}, \quad 0 \leq x \leq a$$

and zero elsewhere. The coefficients B_n are determined by using the provided initial condition,

$$\begin{aligned} \Psi(x, 0) &= A[\psi_1(x) + \psi_2(x)] \\ &= A \left(\sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} + \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} \right). \end{aligned}$$

Before doing so, though, first normalize the initial wave function to find A .

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx \\ &= \int_{-\infty}^{\infty} \Psi(x, 0) \Psi^*(x, 0) dx \end{aligned}$$

Substitute $\Psi(x, 0)$ and evaluate the integral.

$$\begin{aligned}
 1 &= \int_0^a \left[A \sqrt{\frac{2}{a}} \left(\sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right) \right] \left[A \sqrt{\frac{2}{a}} \left(\sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right) \right]^* dx \\
 &= \int_0^a \left[A \sqrt{\frac{2}{a}} \left(\sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right) \right] \left[A \sqrt{\frac{2}{a}} \left(\sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right) \right] dx \\
 &= \frac{2A^2}{a} \int_0^a \left(\sin^2 \frac{\pi x}{a} + 2 \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \sin^2 \frac{2\pi x}{a} \right) dx \\
 &= \frac{2A^2}{a} \left(\int_0^a \sin^2 \frac{\pi x}{a} dx + 2 \int_0^a \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} dx + \int_0^a \sin^2 \frac{2\pi x}{a} dx \right) \\
 &= \frac{2A^2}{a} \left\{ \int_0^a \frac{1}{2} \left(1 - \cos \frac{2\pi x}{a} \right) dx + 2 \int_0^a \frac{1}{2} \left[\cos \left(\frac{\pi x}{a} - \frac{2\pi x}{a} \right) - \cos \left(\frac{\pi x}{a} + \frac{2\pi x}{a} \right) \right] dx \right. \\
 &\quad \left. + \int_0^a \frac{1}{2} \left(1 - \cos \frac{4\pi x}{a} \right) dx \right\} \\
 &= \frac{A^2}{a} \left[\int_0^a \left(1 - \cos \frac{2\pi x}{a} \right) dx + 2 \int_0^a \left(\cos \frac{\pi x}{a} - \cos \frac{3\pi x}{a} \right) dx + \int_0^a \left(1 - \cos \frac{4\pi x}{a} \right) dx \right] \\
 &= \frac{A^2}{a} \left[\left(x - \frac{a}{2\pi} \sin \frac{2\pi x}{a} \right) \Big|_0^a + 2 \left(\frac{a}{\pi} \sin \frac{\pi x}{a} - \frac{a}{3\pi} \sin \frac{3\pi x}{a} \right) \Big|_0^a + \left(x - \frac{a}{4\pi} \sin \frac{4\pi x}{a} \right) \Big|_0^a \right] \\
 &= \frac{A^2}{a} [(a) + 2(0) + (a)] \\
 &= 2A^2
 \end{aligned}$$

Solve for A .

$$A = \frac{1}{\sqrt{2}}$$

With it, the initial condition becomes

$$\Psi(x, 0) = \frac{1}{\sqrt{a}} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} \sin \frac{2\pi x}{a}.$$

Now set $t = 0$ in the general solution.

$$\Psi(x, 0) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} = \sqrt{\frac{2}{a}} B_1 \sin \frac{\pi x}{a} + \sqrt{\frac{2}{a}} B_2 \sin \frac{2\pi x}{a} + \sqrt{\frac{2}{a}} B_3 \sin \frac{3\pi x}{a} + \dots$$

Comparing the coefficients, we see that

$$\begin{aligned}
 \sqrt{\frac{2}{a}} B_1 &= \frac{1}{\sqrt{a}} & \rightarrow & B_1 = \frac{1}{\sqrt{2}} \\
 \sqrt{\frac{2}{a}} B_2 &= \frac{1}{\sqrt{a}} & \rightarrow & B_2 = \frac{1}{\sqrt{2}} \\
 \sqrt{\frac{2}{a}} B_n &= 0, \quad n \geq 3 & \rightarrow & B_n = 0, \quad n \geq 3.
 \end{aligned}$$

Therefore,

$$\begin{aligned}\Psi(x, t) &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \exp\left(-i \frac{\hbar \pi^2 n^2}{2ma^2} t\right) \sin \frac{n\pi x}{a} \\ &= \sqrt{\frac{2}{a}} B_1 \exp\left(-i \frac{\hbar \pi^2 1^2}{2ma^2} t\right) \sin \frac{\pi x}{a} + \sqrt{\frac{2}{a}} B_2 \exp\left(-i \frac{\hbar \pi^2 2^2}{2ma^2} t\right) \sin \frac{2\pi x}{a} \\ &= \frac{1}{\sqrt{a}} \exp\left(-i \frac{\hbar \pi^2}{2ma^2} t\right) \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} \exp\left(-i \frac{2\hbar \pi^2}{ma^2} t\right) \sin \frac{2\pi x}{a},\end{aligned}$$

or using $\omega = \hbar \pi^2 / 2ma^2$ to simplify the result,

$$\Psi(x, t) = \frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a}, \quad 0 \leq x \leq a.$$

Writing the solution in terms of the eigenstates,

$$\begin{aligned}\Psi(x, t) &= \frac{1}{\sqrt{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \right) e^{-i\omega t} + \frac{1}{\sqrt{2}} \left(\sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} \right) e^{-4i\omega t} \\ &= \frac{1}{\sqrt{2}} \psi_1(x) e^{-i\omega t} + \frac{1}{\sqrt{2}} \psi_2(x) e^{-4i\omega t},\end{aligned}$$

we can see that the probabilities of measuring

$$\begin{aligned}\frac{E_1}{\hbar} = \omega &\quad \rightarrow \quad E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \\ \frac{E_2}{\hbar} = 4\omega &\quad \rightarrow \quad E_2 = \frac{2\hbar^2 \pi^2}{ma^2}\end{aligned}$$

are

$$\begin{aligned}P(E_1) &= \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} \\ P(E_2) &= \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},\end{aligned}$$

respectively. The expectation value of the energy is

$$\langle H \rangle = P(E_1)E_1 + P(E_2)E_2 = \left(\frac{1}{2}\right) \frac{\hbar^2 \pi^2}{2ma^2} + \left(\frac{1}{2}\right) \frac{2\hbar^2 \pi^2}{ma^2} = \frac{5\hbar^2 \pi^2}{4ma^2}.$$

The probability distribution for the particle's position at time t is

$$\begin{aligned}|\Psi(x, t)|^2 &= \Psi(x, t)\Psi^*(x, t) \\ &= \left(\frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a} \right) \left(\frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a} \right)^* \\ &= \left(\frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a} \right) \left(\frac{1}{\sqrt{a}} e^{i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{4i\omega t} \sin \frac{2\pi x}{a} \right) \\ &= \frac{1}{a} \sin^2 \frac{\pi x}{a} + \frac{1}{a} e^{3i\omega t} \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \frac{1}{a} e^{-3i\omega t} \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \frac{1}{a} \sin^2 \frac{2\pi x}{a} \\ &= \frac{1}{a} \left[\sin^2 \frac{\pi x}{a} + (e^{3i\omega t} + e^{-3i\omega t}) \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \sin^2 \frac{2\pi x}{a} \right] \\ &= \frac{1}{a} \left(\sin^2 \frac{\pi x}{a} + 2 \cos 3\omega t \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \sin^2 \frac{2\pi x}{a} \right), \quad 0 \leq x \leq a.\end{aligned}$$

Observe that the wave function remains normalized for all t .

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = \frac{1}{a} \left(\underbrace{\int_0^a \sin^2 \frac{\pi x}{a} dx}_{= a/2} + 2 \cos 3\omega t \underbrace{\int_0^a \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} dx}_{= 0} + \underbrace{\int_0^a \sin^2 \frac{2\pi x}{a} dx}_{= a/2} \right) = 1$$

Now calculate the expectation value of x at time t .

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} \Psi^*(x, t)(x)\Psi(x, t) dx \\ &= \int_0^a \left(\frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a} \right)^* (x) \left(\frac{1}{\sqrt{a}} e^{-i\omega t} \sin \frac{\pi x}{a} + \frac{1}{\sqrt{a}} e^{-4i\omega t} \sin \frac{2\pi x}{a} \right) dx \\ &= \frac{1}{a} \int_0^a x \left(\sin^2 \frac{\pi x}{a} + 2 \cos 3\omega t \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} + \sin^2 \frac{2\pi x}{a} \right) dx \\ &= \frac{1}{a} \left(\int_0^a x \sin^2 \frac{\pi x}{a} dx + 2 \cos 3\omega t \int_0^a x \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} dx + \int_0^a x \sin^2 \frac{2\pi x}{a} dx \right) \\ &= \frac{1}{a} \left\{ \int_0^a \frac{x}{2} \left(1 - \cos \frac{2\pi x}{a} \right) dx + 2 \cos 3\omega t \int_0^a \frac{x}{2} \left[\cos \left(\frac{\pi x}{a} - \frac{2\pi x}{a} \right) - \cos \left(\frac{\pi x}{a} + \frac{2\pi x}{a} \right) \right] dx \right. \\ &\quad \left. + \int_0^a \frac{x}{2} \left(1 - \cos \frac{4\pi x}{a} \right) dx \right\} \\ &= \frac{1}{2a} \left[\int_0^a x dx - \int_0^a x \cos \frac{2\pi x}{a} dx + 2 \cos 3\omega t \left(\int_0^a x \cos \frac{\pi x}{a} dx - \int_0^a x \cos \frac{3\pi x}{a} dx \right) \right. \\ &\quad \left. + \int_0^a x dx - \int_0^a x \cos \frac{4\pi x}{a} dx \right] \\ &= \frac{1}{2a} \left[\frac{a^2}{2} - 0 + 2 \cos 3\omega t \left(-\frac{2a^2}{\pi^2} + \frac{2a^2}{9\pi^2} \right) + \frac{a^2}{2} - 0 \right] \\ &= \frac{1}{2a} \left(a^2 - \frac{32a^2}{9\pi^2} \cos 3\omega t \right) \\ &= \frac{a}{2} - \frac{16a}{9\pi^2} \cos 3\omega t \end{aligned}$$

The expectation value of x oscillates in time with an amplitude and angular frequency of

$$\frac{16a}{9\pi^2} \approx 0.180a \quad \text{and} \quad 3\omega = \frac{3\hbar\pi^2}{2ma^2},$$

respectively. Finally, use Ehrenfest's theorem to calculate the expectation value of p at time t .

$$\begin{aligned} \langle p \rangle &= m \frac{d\langle x \rangle}{dt} = m \frac{d}{dt} \left(\frac{a}{2} - \frac{16a}{9\pi^2} \cos 3\omega t \right) = \frac{16ma\omega}{3\pi^2} \sin 3\omega t \\ &= \frac{16ma}{3\pi^2} \left(\frac{\hbar\pi^2}{2ma^2} \right) \sin \left(\frac{3\hbar\pi^2}{2ma^2} t \right) \\ &= \frac{8\hbar}{3a} \sin \left(\frac{3\hbar\pi^2}{2ma^2} t \right) \end{aligned}$$