Problem 2.50

Consider the *moving* delta-function well:

$$V(x,t) = -\alpha\delta(x - vt),$$

where v is the (constant) velocity of the well.

(a) Show that the time-dependent Schrödinger equation admits the exact solution[60]

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x-vt|/\hbar^2} e^{-i\left[(E+(1/2)mv^2)t - mvx\right]/\hbar},$$

where $E = -m\alpha^2/2\hbar^2$ is the bound-state energy of the *stationary* delta function. *Hint:* Plug it in and *check* it! Use the result of Problem 2.23(b).

(b) Find the expectation value of the Hamiltonian in this state, and comment on the result.

Solution

Part (a)

Here the goal is merely to verify that

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x - vt|\right) \exp\left\{-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t - mvx\right]\right\}$$

is a solution to the Schrödinger equation with a moving delta-function potential well.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha\delta(x - vt)\Psi(x,t)$$

The absolute value function is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases} = x\begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} = x \operatorname{sgn} x = x[\theta(x) - \theta(-x)],$$

where $\operatorname{sgn} x$ is the sign (signum) function and $\theta(x)$ is the Heaviside function, which is 1 if $x > 0$ and 0 if $x < 0$. Rewrite Ψ using this more convenient formula.

$$\begin{aligned}\Psi(x,t) &= \frac{\sqrt{m\alpha}}{\hbar} \exp\left\{-\frac{m\alpha}{\hbar^2}(x - vt)[\theta(x - vt) - \theta(vt - x)]\right\} \exp\left\{-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t - mvx\right]\right\} \\ &= \frac{\sqrt{m\alpha}}{\hbar} \exp\left\{-\frac{m\alpha}{\hbar^2}(x - vt)[\theta(x - vt) - \theta(vt - x)] - \frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t - mvx\right]\right\}\end{aligned}$$

[60] See Problem 6.35 for a derivation.

Calculate the first derivative with respect to t.

$$\begin{aligned}
\frac{\partial\Psi}{\partial t} &= \frac{\sqrt{m\alpha}}{\hbar}\exp\left\{-\frac{m\alpha}{\hbar^2}(x-vt)[\theta(x-vt)-\theta(vt-x)]-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t-mvx\right]\right\} \\
&\quad\times\frac{\partial}{\partial t}\left\{-\frac{m\alpha}{\hbar^2}(x-vt)[\theta(x-vt)-\theta(vt-x)]-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t-mvx\right]\right\} \\
&= \Psi(x,t) \\
&\quad\times\left\{-\frac{m\alpha}{\hbar^2}\Big[-v[\theta(x-vt)-\theta(vt-x)]+(x-vt)[-v\theta'(x-vt)-v\theta'(vt-x)]\Big]-\frac{i}{\hbar}\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\} \\
&= \Psi(x,t)\left\{\frac{m\alpha v}{\hbar^2}\Big[\theta(x-vt)-\theta(vt-x)+(x-vt)[\delta(x-vt)+\delta(vt-x)]\Big]-\frac{i}{\hbar}\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\} \\
&= \Psi(x,t)\left\{\frac{m\alpha v}{\hbar^2}\Big[\theta(x-vt)-\theta(vt-x)+\underbrace{2(x-vt)\delta(x-vt)}_{=0}\Big]-\frac{i}{\hbar}\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\} \\
&= \Psi(x,t)\left\{\frac{m\alpha v}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]-\frac{i}{\hbar}\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\}
\end{aligned}$$

Calculate the first derivative with respect to x.

$$\begin{aligned}
\frac{\partial\Psi}{\partial x} &= \frac{\sqrt{m\alpha}}{\hbar}\exp\left\{-\frac{m\alpha}{\hbar^2}(x-vt)[\theta(x-vt)-\theta(vt-x)]-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t-mvx\right]\right\} \\
&\quad\times\frac{\partial}{\partial x}\left\{-\frac{m\alpha}{\hbar^2}(x-vt)[\theta(x-vt)-\theta(vt-x)]-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t-mvx\right]\right\} \\
&= \Psi(x,t) \\
&\quad\times\left\{-\frac{m\alpha}{\hbar^2}\Big[[\theta(x-vt)-\theta(vt-x)]+(x-vt)[\theta'(x-vt)+\theta'(vt-x)]\Big]-\frac{i}{\hbar}(-mv)\right\} \\
&= \Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}\Big[[\theta(x-vt)-\theta(vt-x)]+(x-vt)[\delta(x-vt)+\delta(vt-x)]\Big]+\frac{imv}{\hbar}\right\} \\
&= \Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}\Big[[\theta(x-vt)-\theta(vt-x)]+\underbrace{2(x-vt)\delta(x-vt)}_{=0}\Big]+\frac{imv}{\hbar}\right\} \\
&= \Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]+\frac{imv}{\hbar}\right\}
\end{aligned}$$

Calculate the second derivative with respect to x.

$$\begin{aligned}
\frac{\partial^2\Psi}{\partial x^2} &= \frac{\partial\Psi}{\partial x}\left\{-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]+\frac{imv}{\hbar}\right\}+\Psi(x,t)\frac{\partial}{\partial x}\left\{-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]+\frac{imv}{\hbar}\right\} \\
&= \Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]+\frac{imv}{\hbar}\right\}^2+\Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}[\theta'(x-vt)+\theta'(vt-x)]\right\} \\
&= \Psi(x,t)\left\{\left[-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)]+\frac{imv}{\hbar}\right]^2-\frac{m\alpha}{\hbar^2}[\delta(x-vt)+\delta(vt-x)]\right\}
\end{aligned}$$

Continue the simplification.

$$\frac{\partial^2\Psi}{\partial x^2} = \Psi(x,t)\left\{\left[\frac{m^2\alpha^2}{\hbar^4}\underbrace{[\theta(x-vt)-\theta(vt-x)]^2}_{=1} - \frac{m^2v^2}{\hbar^2} - \frac{2im^2\alpha v}{\hbar^3}[\theta(x-vt)-\theta(vt-x)]\right] - \frac{2m\alpha}{\hbar^2}\delta(x-vt)\right\}$$

$$= \Psi(x,t)\left\{\frac{m^2\alpha^2}{\hbar^4} - \frac{m^2v^2}{\hbar^2} - \frac{2im^2\alpha v}{\hbar^3}[\theta(x-vt)-\theta(vt-x)] - \frac{2m\alpha}{\hbar^2}\delta(x-vt)\right\}$$

Evaluate the left side of Schrödinger's equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\Psi(x,t)\left\{\frac{m\alpha v}{\hbar^2}[\theta(x-vt)-\theta(vt-x)] - \frac{i}{\hbar}\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\}$$

$$= \Psi(x,t)\left\{\frac{im\alpha v}{\hbar}[\theta(x-vt)-\theta(vt-x)] + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\}$$

Evaluate the right side of Schrödinger's equation.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha\delta(x-vt)\Psi(x,t) = -\frac{\hbar^2}{2m}\Psi(x,t)\left\{\frac{m^2\alpha^2}{\hbar^4} - \frac{m^2v^2}{\hbar^2} - \frac{2im^2\alpha v}{\hbar^3}[\theta(x-vt)-\theta(vt-x)] - \frac{2m\alpha}{\hbar^2}\delta(x-vt)\right\}$$
$$- \alpha\delta(x-vt)\Psi(x,t)$$

$$= \Psi(x,t)\left\{-\frac{m\alpha^2}{2\hbar^2} + \frac{mv^2}{2} + \frac{im\alpha v}{\hbar}[\theta(x-vt)-\theta(vt-x)] + \cancel{\alpha\delta(x-vt)}\right\}$$
$$- \cancel{\alpha\delta(x-vt)\Psi(x,t)}$$

$$= \Psi(x,t)\left\{\frac{im\alpha v}{\hbar}[\theta(x-vt)-\theta(vt-x)] + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\}$$

Because both sides of the Schrödinger equation evaluate to the same function, the formula for $\Psi(x,t)$ is indeed a solution.

Part (b)

Calculate the expectation value of the Hamiltonian operator.

$$\begin{aligned}
\langle H\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(\hat{H})\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(\frac{\hat{p}^2}{2m}+V\right)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\Psi^*(x,t)\left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2 - \alpha\delta(x-vt)\right]\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\Psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \alpha\delta(x-vt)\right]\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}\Psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha\delta(x-vt)\Psi(x,t)\right]dx
\end{aligned}$$

Substitute the formula for the right side of Schrödinger's equation.

$$\begin{aligned}\langle H\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\left\{\frac{im\alpha v}{\hbar}[\theta(x-vt)-\theta(vt-x)] + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right\}dx \\ &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2\left[\frac{im\alpha v}{\hbar}\operatorname{sgn}(x-vt) + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\right]dx \\ &= \frac{im\alpha v}{\hbar}\int_{-\infty}^{\infty}\operatorname{sgn}(x-vt)|\Psi(x,t)|^2\,dx + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)\underbrace{\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx}_{=\,1}\end{aligned}$$

The probability distribution for the particle's position at time t is

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\ &= \frac{\sqrt{m\alpha}}{\hbar}\exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right)\exp\left\{-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t - mvx\right]\right\} \\ &\quad\times\frac{\sqrt{m\alpha}}{\hbar}\exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right)\exp\left\{\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right)t - mvx\right]\right\} \\ &= \frac{m\alpha}{\hbar^2}\exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right).\end{aligned}$$

The expectation value then becomes

$$\langle H\rangle = \frac{im\alpha v}{\hbar}\int_{-\infty}^{\infty}\operatorname{sgn}(x-vt)\frac{m\alpha}{\hbar^2}\exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right)dx + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right).$$

Make the following substitution in the integral.

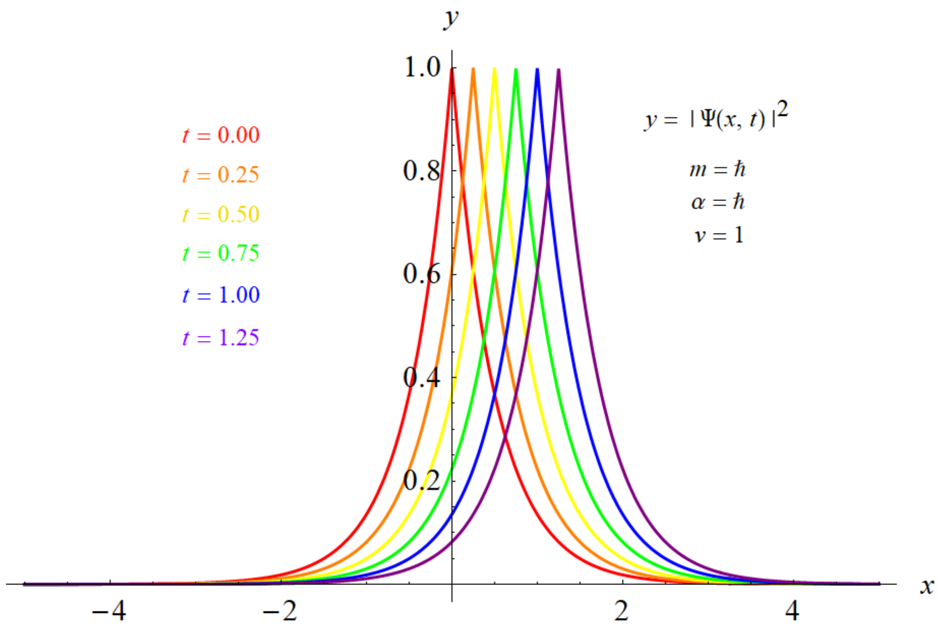
$$\begin{aligned} r &= x-vt \\ dr &= dx\end{aligned}$$

Therefore,

$$\begin{aligned}\langle H\rangle &= \frac{im\alpha v}{\hbar}\underbrace{\int_{-\infty}^{\infty}(\operatorname{sgn} r)\frac{m\alpha}{\hbar^2}\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)dr}_{=\,0} + \left(-\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2\right) \\ &= -\frac{m\alpha^2}{2\hbar^2}+\frac{1}{2}mv^2,\end{aligned}$$

since the integrand is an odd function. This is the energy of the stationary delta-function well plus the kinetic energy of motion.

Below is a plot of the probability distribution versus x at various times for the special case that $m = \hbar$, $\alpha = \hbar$, and $v = 1$.



Part (c)

Calculate the expectation value of x at time t.

$$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx = \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty} x\exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right)dx$$

Make the following substitution.

$$\begin{aligned} r &= x - vt \quad \to \quad x = r + vt \\ dr &= dx \end{aligned}$$

Consequently,

$$\begin{aligned} \langle x\rangle &= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty}(r+vt)\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)dr \\ &= \frac{m\alpha}{\hbar^2}\left[\underbrace{\int_{-\infty}^{\infty} r\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)dr}_{=\,0} + vt\int_{-\infty}^{\infty}\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)dr\right] \\ &= \frac{m\alpha}{\hbar^2}vt\int_{-\infty}^{\infty}\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)dr \\ &= 2\frac{m\alpha}{\hbar^2}vt\int_0^{\infty}\exp\left(-\frac{2m\alpha}{\hbar^2}r\right)dr \\ &= 2\frac{m\alpha}{\hbar^2}vt\left(\frac{\hbar^2}{2m\alpha}\right) \\ &= vt. \end{aligned}$$

Calculate the expectation value of x^2 at time t.

$$\begin{aligned}\langle x^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\ &= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2\,dx \\ &= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right) dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned} r &= x - vt \quad \rightarrow \quad x = r + vt \\ dr &= dx \end{aligned}$$

Consequently,

$$\begin{aligned}\langle x^2\rangle &= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty}(r+vt)^2 \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr \\ &= \frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty}(r^2+2vtr+v^2t^2) \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr \\ &= \frac{m\alpha}{\hbar^2}\left[\int_{-\infty}^{\infty} r^2 \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr + 2vt\underbrace{\int_{-\infty}^{\infty} r \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr}_{=0} + v^2t^2\int_{-\infty}^{\infty} \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr\right] \\ &= \frac{m\alpha}{\hbar^2}\left[2\int_0^{\infty} r^2 \exp\left(-\frac{2m\alpha}{\hbar^2}r\right) dr + 2v^2t^2\int_0^{\infty} \exp\left(-\frac{2m\alpha}{\hbar^2}r\right) dr\right] \\ &= \frac{m\alpha}{\hbar^2}\left[2\cdot 2\left(\frac{\hbar^2}{2m\alpha}\right)^3 + 2v^2t^2\left(\frac{\hbar^2}{2m\alpha}\right)\right] \\ &= \frac{\hbar^4}{2m^2\alpha^2} + v^2t^2.\end{aligned}$$

The standard deviation in x is then

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(\frac{\hbar^4}{2m^2\alpha^2} + v^2t^2\right) - v^2t^2} = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

According to Ehrenfest's theorem,

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}(vt) = mv.$$

Check this result by calculating the expectation value of p at time t directly.

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\ &= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial\Psi}{\partial x}\,dx \\ &= -i\hbar\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\left\{-\frac{m\alpha}{\hbar^2}[\theta(x-vt)-\theta(vt-x)] + \frac{imv}{\hbar}\right\} dx\end{aligned}$$

Continue the simplification.

$$\begin{aligned}\langle p\rangle &= -i\hbar\int_{-\infty}^{\infty}|\Psi(x,t)|^2\left[-\frac{m\alpha}{\hbar^2}\operatorname{sgn}(x-vt)+\frac{imv}{\hbar}\right]dx\\ &= -i\hbar\Bigg[-\frac{m\alpha}{\hbar^2}\underbrace{\int_{-\infty}^{\infty}\operatorname{sgn}(x-vt)|\Psi(x,t)|^2\,dx}_{=0}+\frac{imv}{\hbar}\underbrace{\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx}_{=1}\Bigg]\\ &= -i\hbar\left(\frac{imv}{\hbar}\right)\\ &= mv\end{aligned}$$

Now calculate the expectation value of p^2 at time t.

$$\begin{aligned}\langle p^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx\\ &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\Psi(x,t)\,dx\\ &= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial^2\Psi}{\partial x^2}\,dx\\ &= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\Psi(x,t)\left\{\frac{m^2\alpha^2}{\hbar^4}-\frac{m^2v^2}{\hbar^2}-\frac{2im^2\alpha v}{\hbar^3}[\theta(x-vt)-\theta(vt-x)]-\frac{2m\alpha}{\hbar^2}\delta(x-vt)\right\}dx\\ &= \int_{-\infty}^{\infty}|\Psi(x,t)|^2\left[-\frac{m^2\alpha^2}{\hbar^2}+m^2v^2+\frac{2im^2\alpha v}{\hbar}\operatorname{sgn}(x-vt)+2m\alpha\delta(x-vt)\right]dx\\ &= \left(-\frac{m^2\alpha^2}{\hbar^2}+m^2v^2\right)\overbrace{\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx}^{=1}+\frac{2im^2\alpha v}{\hbar}\overbrace{\int_{-\infty}^{\infty}\operatorname{sgn}(x-vt)|\Psi(x,t)|^2\,dx}^{=0}\\ &\qquad +2m\alpha\int_{-\infty}^{\infty}|\Psi(x,t)|^2\delta(x-vt)\,dx\\ &= \left(-\frac{m^2\alpha^2}{\hbar^2}+m^2v^2\right)+2m\alpha\int_{-\infty}^{\infty}\frac{m\alpha}{\hbar^2}\exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right)\delta(x-vt)\,dx\\ &= \left(-\frac{m^2\alpha^2}{\hbar^2}+m^2v^2\right)+\frac{2m^2\alpha^2}{\hbar^2}e^0\\ &= \frac{m^2\alpha^2}{\hbar^2}+m^2v^2\end{aligned}$$

The standard deviation in p is then

$$\sigma_p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\left(\frac{m^2\alpha^2}{\hbar^2}+m^2v^2\right)-m^2v^2}=\frac{m\alpha}{\hbar}.$$

Therefore, the uncertainty product is consistent with Heisenberg's principle ($\sigma_x\sigma_p\geq\hbar/2$).

$$\sigma_x\sigma_p=\frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar}=\frac{\hbar}{\sqrt{2}}\approx 0.707\hbar$$

Finally, check to see if Equation (1.38) on page 18 is satisfied.

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle \tag{1.38}$$

The left-hand side evaluates to

$$\frac{d\langle p\rangle}{dt} = \frac{d}{dt}(mv) = 0,$$

and the right-hand side evaluates to

$$\begin{aligned}
\left\langle -\frac{\partial V}{\partial x}\right\rangle &= \left\langle -\frac{\partial}{\partial x}[-\alpha\delta(x-vt)]\right\rangle \\
&= \langle \alpha\delta'(x-vt)\rangle \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t)[\alpha\delta'(x-vt)]\Psi(x,t)\,dx \\
&= \alpha\int_{-\infty}^{\infty} |\Psi(x,t)|^2\delta'(x-vt)\,dx \\
&= \alpha\int_{-\infty}^{\infty} \frac{m\alpha}{\hbar^2}\exp\left(-\frac{2m\alpha}{\hbar^2}|x-vt|\right)\delta'(x-vt)\,dx.
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
r &= x - vt \\
dr &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\langle -\frac{\partial V}{\partial x}\right\rangle &= \frac{m\alpha^2}{\hbar^2}\int_{-\infty}^{\infty}\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)\delta'(r)\,dr \\
&= \frac{m\alpha^2}{\hbar^2}\Bigg\{\underbrace{\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)\delta(r)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}\left[\frac{d}{dr}\exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right)\right]\delta(r)\,dr\Bigg\} \\
&= -\frac{m\alpha^2}{\hbar^2}\int_{-\infty}^{\infty}\delta(r)\frac{d}{dr}\exp\left\{-\frac{2m\alpha}{\hbar^2}r[\theta(r)-\theta(-r)]\right\}dr \\
&= -\frac{m\alpha^2}{\hbar^2}\int_{-\infty}^{\infty}\delta(r)\exp\left\{-\frac{2m\alpha}{\hbar^2}r[\theta(r)-\theta(-r)]\right\}\left(-\frac{2m\alpha}{\hbar^2}\right)\left\{[\theta(r)-\theta(-r)] + r[\theta'(r)+\theta'(-r)]\right\}dr \\
&= \frac{2m^2\alpha^3}{\hbar^4}\int_{-\infty}^{\infty}\delta(r)\exp\left\{-\frac{2m\alpha}{\hbar^2}r[\theta(r)-\theta(-r)]\right\}\left\{[\theta(r)-\theta(-r)] + r[\delta(r)+\delta(-r)]\right\}dr \\
&= \frac{2m^2\alpha^3}{\hbar^4}\int_{-\infty}^{\infty}\delta(r)\exp\left\{-\frac{2m\alpha}{\hbar^2}r[\theta(r)-\theta(-r)]\right\}\Big\{[\theta(r)-\theta(-r)] + \underbrace{2r\delta(r)}_{=0}\Big\}\,dr \\
&= \frac{2m^2\alpha^3}{\hbar^4}e^0[\theta(0)-\theta(0)] = 0,
\end{aligned}$$

which means Equation 1.38 is satisfied.

Part (d)

Here the solution given in part (a) will be derived. The aim is to solve the Schrödinger equation subject to the usual boundary conditions, $\Psi \to 0$ as $x \to \pm\infty$, with a moving delta-function potential well.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \alpha\delta(x - vt)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

Despite the fact that this PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables will not work because the delta function has both x and t in its argument. Make the following change of variables then.

$$r = x - vt \qquad s = t$$

Use the chain rule to write the derivatives of Ψ in terms of these new variables.

$$\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial \Psi}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial \Psi}{\partial r}(-v) + \frac{\partial \Psi}{\partial s}(1) = -v\frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial s}$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Psi}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial \Psi}{\partial r}(1) + \frac{\partial \Psi}{\partial s}(0) = \frac{\partial \Psi}{\partial r}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial x}\right) = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s}\right)\left(\frac{\partial \Psi}{\partial r}\right) = \left(\frac{\partial}{\partial r}\right)\left(\frac{\partial \Psi}{\partial r}\right) = \frac{\partial^2 \Psi}{\partial r^2}$$

Substitute these formulas into the PDE.

$$i\hbar\left(-v\frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial s}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial r^2} - \alpha\delta(r)\Psi(r,s)$$

Bring the derivative with respect to r to the right side.

$$i\hbar\frac{\partial \Psi}{\partial s} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial r^2} + i\hbar v\frac{\partial \Psi}{\partial r} - \alpha\delta(r)\Psi(r,s), \quad -\infty < r < \infty,\ s > 0$$

At the cost of an extra linear term on the right side, the delta function has been made a function of r only. Now apply the method of separation of variables: Assume a product solution of the form $\Psi(r,s) = \psi(r)\phi(s)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial s}[\psi(r)\phi(s)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}[\psi(r)\phi(s)] + i\hbar v\frac{\partial}{\partial r}[\psi(r)\phi(s)] - \alpha\delta(r)[\psi(r)\phi(s)]$$

Evaluate the derivatives.

$$i\hbar\psi(r)\phi'(s) = -\frac{\hbar^2}{2m}\psi''(r)\phi(s) + i\hbar v\psi'(r)\phi(s) - \alpha\delta(r)\psi(r)\phi(s)$$

Divide both sides by $\psi(r)\phi(s)$ to separate variables.

$$i\hbar\frac{\phi'(s)}{\phi(s)} = -\frac{\hbar^2}{2m}\frac{\psi''(r)}{\psi(r)} + i\hbar v\frac{\psi'(r)}{\psi(r)} - \alpha\delta(r)$$

The only way a function of s can be equal to a function of r is if both are equal to a constant.

$$i\hbar\frac{\phi'(s)}{\phi(s)} = -\frac{\hbar^2}{2m}\frac{\psi''(r)}{\psi(r)} + i\hbar v\frac{\psi'(r)}{\psi(r)} - \alpha\delta(r) = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs, one in r and one in s.

$$\left.\begin{aligned} i\hbar\frac{\phi'(s)}{\phi(s)} &= \lambda \\ -\frac{\hbar^2}{2m}\frac{\psi''(r)}{\psi(r)} + i\hbar v\frac{\psi'(r)}{\psi(r)} - \alpha\delta(r) &= \lambda \end{aligned}\right\}$$

The ODE in r can be written as

$$\frac{d^2\psi}{dr^2} - \frac{2imv}{\hbar}\frac{d\psi}{dr} + \frac{2m}{\hbar^2}[\alpha\delta(r) + \lambda]\psi(r) = 0. \tag{1}$$

For points not at $r = 0$, the delta function vanishes.

$$\frac{d^2\psi}{dr^2} - \frac{2imv}{\hbar}\frac{d\psi}{dr} + \frac{2m\lambda}{\hbar^2}\psi(r) = 0, \quad r \neq 0$$

Since this is an ODE with constant coefficients, its solutions are of the form $\psi(r) = e^{pr}$, where p is to be determined.

$$\psi = e^{pr} \quad \rightarrow \quad \frac{d\psi}{dr} = pe^{pr} \quad \rightarrow \quad \frac{d^2\psi}{dr^2} = p^2e^{pr}$$

Substitute these formulas into the ODE.

$$p^2e^{pr} - \frac{2imv}{\hbar}pe^{pr} + \frac{2m\lambda}{\hbar^2}e^{pr} = 0$$

Multiply both sides by $\hbar^2 e^{-pr}$ and then solve the resulting equation for p.

$$\hbar^2p^2 - 2i\hbar mvp + 2m\lambda = 0$$

$$p = \frac{2i\hbar mv \pm \sqrt{(2i\hbar mv)^2 - 8m\hbar^2\lambda}}{2\hbar^2} = \frac{imv \pm \sqrt{-m^2v^2 - 2m\lambda}}{\hbar}$$

For p to have a real part (that is, for there to be a bound state), the eigenvalue λ must be negative: $\lambda = -\gamma^2$.

$$p = \frac{imv \pm \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}$$

The general solution for ψ is then

$$\psi(r) = \begin{cases} C_1 \exp\left(\dfrac{imv + \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) + C_2 \exp\left(\dfrac{imv - \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) & \text{if } r < 0 \\ C_3 \exp\left(\dfrac{imv + \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) + C_4 \exp\left(\dfrac{imv - \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) & \text{if } r > 0 \end{cases}.$$

To prevent ψ from blowing up as $r \to -\infty$ and $r \to \infty$, set $C_2 = 0$ and $C_3 = 0$, respectively.

$$\psi(r) = \begin{cases} C_1 \exp\left(\dfrac{imv + \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) & \text{if } r < 0 \\ C_4 \exp\left(\dfrac{imv - \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}r\right) & \text{if } r > 0 \end{cases}$$

The wave function is required to be continuous at all points, including $r = 0$, so

$$\lim_{r\to 0^-} \psi(r) = \lim_{r\to 0^+} \psi(r) : \quad C_1 = C_4.$$

As a result,

$$\psi(r) = \begin{cases} C_1 \exp\left(\dfrac{imv + \sqrt{2m\gamma^2 - m^2v^2}}{\hbar} r\right) & \text{if } r < 0 \\ C_1 \exp\left(\dfrac{imv - \sqrt{2m\gamma^2 - m^2v^2}}{\hbar} r\right) & \text{if } r > 0 \end{cases}.$$

To determine γ, integrate both sides of equation (1) with respect to r from $-\epsilon$ to ϵ, where ϵ is a really small positive number.

$$\begin{aligned} 0 &= \int_{-\epsilon}^{\epsilon} \left\{ \frac{d^2\psi}{dr^2} - \frac{2imv}{\hbar}\frac{d\psi}{dr} + \frac{2m}{\hbar^2}[\alpha\delta(r) + \lambda]\psi(r) \right\} dr \\ &= \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dr^2}\, dr - \frac{2imv}{\hbar} \int_{-\epsilon}^{\epsilon} \frac{d\psi}{dr}\, dr + \frac{2m}{\hbar^2}\left[\alpha \int_{-\epsilon}^{\epsilon} \psi(r)\delta(r)\, dr + \lambda \int_{-\epsilon}^{\epsilon} \psi(r)\, dr\right] \\ &= \left.\frac{d\psi}{dr}\right|_{-\epsilon}^{\epsilon} - \frac{2imv}{\hbar}\psi\Big|_{-\epsilon}^{\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \lambda\psi(0)\int_{-\epsilon}^{\epsilon} dr\right] \\ &= \left.\frac{d\psi}{dr}\right|_{-\epsilon}^{\epsilon} - \frac{2imv}{\hbar}\psi\Big|_{-\epsilon}^{\epsilon} + \frac{2m}{\hbar^2}\left[\alpha\psi(0) + \lambda\psi(0)(2\epsilon)\right] \end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\begin{aligned} 0 &= \left.\frac{d\psi}{dr}\right|_{0^-}^{0^+} - \frac{2imv}{\hbar}\psi\Big|_{0^-}^{0^+} + \frac{2m\alpha}{\hbar^2}\psi(0) \\ &= C_1\left(\frac{imv - \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}\right) - C_1\left(\frac{imv + \sqrt{2m\gamma^2 - m^2v^2}}{\hbar}\right) - \frac{2imv}{\hbar}(C_1 - C_1) + \frac{2m\alpha}{\hbar^2}C_1 \\ &= -\frac{2\sqrt{2m\gamma^2 - m^2v^2}}{\hbar}C_1 + \frac{2m\alpha}{\hbar^2}C_1 \end{aligned}$$

To avoid the trivial solution, assume that $C_1 \neq 0$ and solve for γ^2.

$$\sqrt{2m\gamma^2 - m^2v^2} = \frac{m\alpha}{\hbar}$$

$$2m\gamma^2 = \frac{m^2\alpha^2}{\hbar^2} + m^2v^2$$

$$\gamma^2 = \frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2$$

Consequently, the solution for ψ becomes

$$\psi(r) = \begin{cases} C_1 \exp\left(\dfrac{imv + \frac{m\alpha}{\hbar}}{\hbar} r\right) & \text{if } r < 0 \\ C_1 \exp\left(\dfrac{imv - \frac{m\alpha}{\hbar}}{\hbar} r\right) & \text{if } r > 0 \end{cases} = C_1 \exp\left(\frac{imv}{\hbar} r\right) \exp\left(-\frac{m\alpha}{\hbar^2}|r|\right).$$

The constant C_1 is arbitrary and is chosen so that the integral of $|\psi(r)|^2$ over the whole line is 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(r)|^2 \, dr \\
&= \int_{-\infty}^{\infty} \psi(r)\psi^*(r) \, dr \\
&= \int_{-\infty}^{\infty} C_1 \exp\left(\frac{imv}{\hbar}r\right) \exp\left(-\frac{m\alpha}{\hbar^2}|r|\right) C_1 \exp\left(-\frac{imv}{\hbar}r\right) \exp\left(-\frac{m\alpha}{\hbar^2}|r|\right) dr \\
&= C_1^2 \int_{-\infty}^{\infty} \exp\left(-\frac{2m\alpha}{\hbar^2}|r|\right) dr \\
&= 2C_1^2 \int_0^{\infty} \exp\left(-\frac{2m\alpha}{\hbar^2}r\right) dr \\
&= 2C_1^2 \left(\frac{\hbar^2}{2m\alpha}\right)
\end{aligned}$$

Solve for C_1.

$$C_1 = \frac{\sqrt{m\alpha}}{\hbar}$$

The solution to the ODE in s with $\lambda = -\gamma^2$ is

$$\phi(s) = e^{-i\lambda s/\hbar} = \exp\left[\frac{i}{\hbar}\left(\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)s\right].$$

The product solution is then

$$\begin{aligned}
\Psi(r,s) &= \psi(r)\phi(s) \\
&= \frac{\sqrt{m\alpha}}{\hbar} \exp\left(\frac{imv}{\hbar}r\right) \exp\left(-\frac{m\alpha}{\hbar^2}|r|\right) \exp\left[\frac{i}{\hbar}\left(\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)s\right].
\end{aligned}$$

Now that the wave function is known, change back to the original variables.

$$\begin{aligned}
\Psi(x,t) &= \frac{\sqrt{m\alpha}}{\hbar} \exp\left[\frac{imv}{\hbar}(x-vt)\right] \exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right) \exp\left[\frac{i}{\hbar}\left(\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t\right] \\
&= \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right) \exp\left\{\frac{i}{\hbar}\left[\left(\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t + mvx - mv^2t\right]\right\} \\
&= \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right) \exp\left\{\frac{i}{\hbar}\left[\left(\frac{m\alpha^2}{2\hbar^2} - \frac{1}{2}mv^2\right)t + mvx\right]\right\} \\
&= \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x-vt|\right) \exp\left\{-\frac{i}{\hbar}\left[\left(-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2\right)t - mvx\right]\right\}
\end{aligned}$$

Therefore, setting $E = -m\alpha^2/(2\hbar^2)$,

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x-vt|/\hbar^2} e^{-i\left[(E+(1/2)mv^2)t - mvx\right]/\hbar}.$$