Problem 2.51

Free fall. Show that

$$\Psi(x,t) = \Psi_0\left(x + \frac{1}{2}gt^2, t\right)\exp\left[-i\frac{mgt}{\hbar}\left(x + \frac{1}{6}gt^2\right)\right] \tag{2.176}$$

satisfies the time-dependent Schrödinger equation for a particle in a uniform gravitational field,

$$V(x) = mgx, \tag{2.177}$$

where $\Psi_0(x,t)$ is the free gaussian wave packet (Equation 2.111). Find $\langle x\rangle$ as a function of time, and comment on the result.[61]

Solution

From Problem 2.21 on page 61, Equation 2.111 gives the formula for $\Psi_0(x,t)$.

$$\Psi_0(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)$$

The final result is then

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[-\frac{a\left(x+\frac{1}{2}gt^2\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[-i\frac{mgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right].$$

Note that $\Psi_0(x,t)$ is the solution to the Schrödinger equation for a free particle.

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2}$$

Part (a)

The goal here is merely to verify that

$$\begin{aligned}\Psi(x,t) &= \Psi_0\left(x+\frac{1}{2}gt^2,t\right)\exp\left[-i\frac{mgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right]\\ &= \Psi_0\left(x+\frac{1}{2}gt^2,t\right)\exp\left\{-i\frac{mgt}{\hbar}\left[\left(x+\frac{1}{2}gt^2\right)-\frac{1}{3}gt^2\right]\right\}\end{aligned}$$

satisfies the Schrödinger equation with a gravitational potential energy.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mgx\Psi(x,t)$$

Actually, to simplify the procedure, make the following change of variables.

$$r = x + \frac{1}{2}gt^2 \qquad s = t$$

[61] For illuminating discussion see M. Nauenberg, *Am. J. Phys.* **84**, 879 (2016).

Use the chain rule to write the derivatives of Ψ in terms of these new variables.

$$\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial \Psi}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial \Psi}{\partial r}(gt) + \frac{\partial \Psi}{\partial s}(1) = gs\frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial s}$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Psi}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial \Psi}{\partial r}(1) + \frac{\partial \Psi}{\partial s}(0) = \frac{\partial \Psi}{\partial r}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial x}\right) = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s}\right)\left(\frac{\partial \Psi}{\partial r}\right) = \left(\frac{\partial}{\partial r}\right)\left(\frac{\partial \Psi}{\partial r}\right) = \frac{\partial^2 \Psi}{\partial r^2}$$

Substitute these formulas into the Schrödinger equation,

$$i\hbar\left(gs\frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial s}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial r^2} + mg\left(r - \frac{1}{2}gs^2\right)\Psi(r,s), \tag{1}$$

and the solution.

$$\Psi(r,s) = \Psi_0(r,s)\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right]$$

Calculate the first derivative with respect to s.

$$\begin{aligned}
\frac{\partial \Psi}{\partial s} &= \frac{\partial \Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \Psi_0(r,s)\frac{\partial}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] \\
&= \frac{\partial \Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \Psi_0(r,s)\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right]\left(-i\frac{mgr}{\hbar} + i\frac{mg^2s^2}{\hbar}\right) \\
&= \frac{\partial \Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \Psi(r,s)\left(-i\frac{mgr}{\hbar} + i\frac{mg^2s^2}{\hbar}\right)
\end{aligned}$$

Calculate the first derivative with respect to r.

$$\begin{aligned}
\frac{\partial \Psi}{\partial r} &= \frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \Psi_0(r,s)\frac{\partial}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] \\
&= \frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \Psi_0(r,s)\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right]\left(-i\frac{mgs}{\hbar}\right) \\
&= \frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] - i\frac{mgs}{\hbar}\Psi(r,s)
\end{aligned}$$

Calculate the second derivative with respect to r.

$$\begin{aligned}
\frac{\partial^2 \Psi}{\partial r^2} &= \frac{\partial^2 \Psi_0}{\partial r^2}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \frac{\partial \Psi_0}{\partial r}\frac{\partial}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] - i\frac{mgs}{\hbar}\frac{\partial \Psi}{\partial r} \\
&= \frac{\partial^2 \Psi_0}{\partial r^2}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] + \frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right]\left(-i\frac{mgs}{\hbar}\right) \\
&\qquad - i\frac{mgs}{\hbar}\left\{\frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] - i\frac{mgs}{\hbar}\Psi(r,s)\right\} \\
&= \frac{\partial^2 \Psi_0}{\partial r^2}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] - 2i\frac{mgs}{\hbar}\frac{\partial \Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r - \frac{1}{3}gs^2\right)\right] - \frac{m^2g^2s^2}{\hbar^2}\Psi(r,s)
\end{aligned}$$

Evaluate the left-hand side of equation (1).

$$i\hbar\left(gs\frac{\partial\Psi}{\partial r}+\frac{\partial\Psi}{\partial s}\right)=i\hbar\Bigg\{gs\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]-i\frac{mg^2s^2}{\hbar}\Psi(r,s)$$

$$+\frac{\partial\Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]+\Psi(r,s)\left(-i\frac{mgr}{\hbar}+i\frac{mg^2s^2}{\hbar}\right)\Bigg\}$$

$$=i\hbar gs\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]+i\hbar\frac{\partial\Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]+mgr\Psi(r,s)$$

Evaluate the right-hand side of equation (1).

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial r^2}+mg\left(r-\frac{1}{2}gs^2\right)\Psi(r,s)=-\frac{\hbar^2}{2m}\Bigg\{\frac{\partial^2\Psi_0}{\partial r^2}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]$$

$$-2i\frac{mgs}{\hbar}\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]$$

$$-\frac{m^2g^2s^2}{\hbar^2}\Psi(r,s)\Bigg\}-\frac{mg^2s^2}{2}\Psi(r,s)+mgr\Psi(r,s)$$

$$=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial r^2}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]$$

$$+i\hbar gs\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]+mgr\Psi(r,s)$$

$$=i\hbar\frac{\partial\Psi_0}{\partial s}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]$$

$$+i\hbar gs\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgs}{\hbar}\left(r-\frac{1}{3}gs^2\right)\right]+mgr\Psi(r,s)$$

Because the left side and right side of equation (1) evaluate to the same function, the formula for $\Psi(x,t)$ is indeed a solution to Schrödinger's equation with a gravitational potential energy.

Part (b)

The probability distribution for the particle's position at time t is given by

$$|\Psi(x,t)|^2=\Psi(x,t)\Psi^*(x,t)$$

$$=\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left[-\frac{a\left(x+\frac{1}{2}gt^2\right)^2}{1+\frac{2i\hbar at}{m}}\right]\exp\left[-i\frac{mgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right]$$

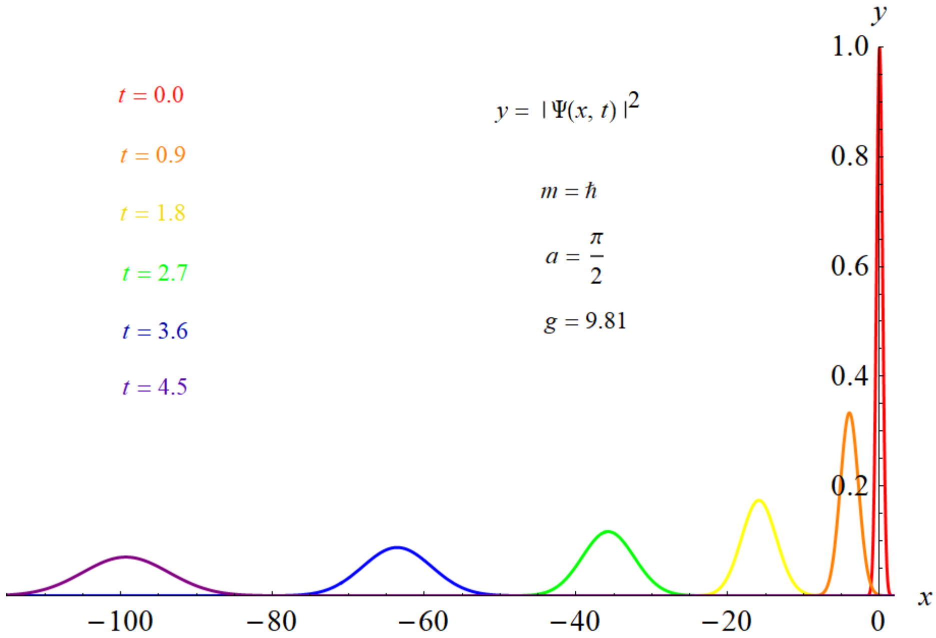
$$\times\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left[-\frac{a\left(x+\frac{1}{2}gt^2\right)^2}{1-\frac{2i\hbar at}{m}}\right]\exp\left[i\frac{mgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right]$$

$$=\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}}\exp\left[-a\left(x+\frac{1}{2}gt^2\right)^2\left(\frac{1}{1+\frac{2i\hbar at}{m}}+\frac{1}{1-\frac{2i\hbar at}{m}}\right)\right].$$

Therefore,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}} \exp\left[-a\left(x+\frac{1}{2}gt^2\right)^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right].$$

Below is an illustration of the probability distribution's time evolution for the special case that $m = \hbar$, $a = \pi/2$, and $g = 9.81$.



Based on this graph, a particle falls with increasing velocity as in the classical scenario, but its position becomes more and more uncertain as time goes on.

Part (c)

Calculate the expectation value of x at time t.

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty} x\sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}} \exp\left[-a\left(x+\frac{1}{2}gt^2\right)^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right] dx
\end{aligned}$$

Make the following substitution.

$$r = x + \frac{1}{2}gt^2 \quad \rightarrow \quad x = r - \frac{1}{2}gt^2$$

$$dr = dx$$

Consequently,

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\left(r-\frac{1}{2}gt^2\right)\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left\{\underbrace{\int_{-\infty}^{\infty} r\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr}_{=0}-\frac{1}{2}gt^2\int_{-\infty}^{\infty}\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr\right\} \\
&= -\sqrt{\frac{2a}{\pi}}\frac{gt^2}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_0^{\infty}\exp\left[-\frac{r^2}{\left(\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}\right)^2}\right]dr \\
&= -\sqrt{\frac{2a}{\pi}}\frac{gt^2}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}}{2}\right) \\
&= -\frac{1}{2}gt^2
\end{aligned}$$

This is the classical result for a falling particle's position. Now calculate the expectation value of x^2 at time t.

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_{-\infty}^{\infty}x^2|\Psi(x,t)|^2\,dx \\
&= \int_{-\infty}^{\infty}x^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\exp\left[-a\left(x+\frac{1}{2}gt^2\right)^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dx
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
r &= x+\frac{1}{2}gt^2 \quad\rightarrow\quad x = r-\frac{1}{2}gt^2 \\
dr &= dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle x^2\rangle &= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\left(r-\frac{1}{2}gt^2\right)^2\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr \\
&= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\left(r^2-gt^2r+\frac{1}{4}g^2t^4\right)\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr.
\end{aligned}$$

Proceed to evaluate the integral.

$$\langle x^2\rangle = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left\{\int_{-\infty}^{\infty} r^2 \exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr - gt^2\overbrace{\int_{-\infty}^{\infty} r \exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr}^{=0}\right.$$
$$\left.+\frac{1}{4}g^2t^4\int_{-\infty}^{\infty}\exp\left[-ar^2\left(\frac{2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)\right]dr\right\}$$
$$= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left\{2\int_0^{\infty} r^2\exp\left[-\frac{r^2}{\left(\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}\right)^2}\right]dr + \frac{1}{2}g^2t^4\int_0^{\infty}\exp\left[-\frac{r^2}{\left(\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}\right)^2}\right]dr\right\}$$
$$= \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\left\{2\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}}{2}\right)^3 + \frac{1}{2}g^2t^4\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}}{2}\right)\right\}$$
$$= \frac{1}{4a} + \frac{\hbar^2at^2}{m^2} + \frac{1}{4}g^2t^4$$

The standard deviation in x is then

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(\frac{1}{4a}+\frac{\hbar^2at^2}{m^2}+\frac{1}{4}g^2t^4\right)-\frac{1}{4}g^2t^4} = \sqrt{\frac{1}{4a}+\frac{\hbar^2at^2}{m^2}}.$$

According to Ehrenfest's theorem, the expectation value of p at time t is

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}\left(-\frac{1}{2}gt^2\right) = -mgt.$$

Check this result by calculating the expectation value directly.

$$\langle p\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$$
$$= -i\hbar\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial\Psi}{\partial x}\,dx$$

Make the following change of variables.

$$r = x + \frac{1}{2}gt^2$$
$$dr = dx$$

Consequently, using the formulas from part (a),

$$\begin{aligned}
\langle p\rangle &= -i\hbar\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\frac{\partial\Psi}{\partial r}\,dr \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\left\{\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]-i\frac{mgt}{\hbar}\Psi(r,t)\right\}dr \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\left\{\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\right. \\
&\qquad\left.-i\frac{mgt}{\hbar}\Psi_0(r,t)\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\right\}dr \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi_0^*(r,t)\left[\frac{\partial\Psi_0}{\partial r}-i\frac{mgt}{\hbar}\Psi_0(r,t)\right]dr \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi_0^*(r,t)\frac{\partial\Psi_0}{\partial r}\,dr - mgt\underbrace{\int_{-\infty}^{\infty}|\Psi_0(r,t)|^2\,dr}_{=1} \\
&= -i\hbar\int_{-\infty}^{\infty}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left(-\frac{ar^2}{1-\frac{2i\hbar at}{m}}\right)\right\}\frac{\partial}{\partial r}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ar^2}{1+\frac{2i\hbar at}{m}}\right)\right\}dr \\
&\qquad - mgt \\
&= -i\hbar\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{ar^2}{1-\frac{2i\hbar at}{m}}\right)\left[\exp\left(-\frac{ar^2}{1+\frac{2i\hbar at}{m}}\right)\left(-\frac{2ar}{1+\frac{2i\hbar at}{m}}\right)\right]dr - mgt \\
&= i\hbar\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\frac{2a}{1+\frac{2i\hbar at}{m}}\underbrace{\int_{-\infty}^{\infty}r\exp\left[-ar^2\left(\frac{1}{1-\frac{2i\hbar at}{m}}+\frac{1}{1+\frac{2i\hbar at}{m}}\right)\right]dr}_{=0} - mgt \\
&= -mgt,
\end{aligned}$$

which confirms Ehrenfest's theorem. Now calculate the expectation value of p^2 at time t.

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi(x,t)\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\frac{\partial^2\Psi}{\partial x^2}\,dx
\end{aligned}$$

Make the following change of variables.

$$\begin{aligned}
r &= x+\frac{1}{2}gt^2 \\
dr &= dx
\end{aligned}$$

Consequently, using the formulas from part (a),

$$\begin{aligned}
\langle p^2\rangle &= -\hbar^2\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\frac{\partial^2\Psi}{\partial r^2}\,dr \\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\left\{\frac{\partial^2\Psi_0}{\partial r^2}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\right. \\
&\qquad -2i\frac{mgt}{\hbar}\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right] \\
&\qquad \left.-\frac{m^2g^2t^2}{\hbar^2}\Psi(r,t)\right\}dr \\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi_0^*(r,t)\exp\left[i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\left\{\frac{\partial^2\Psi_0}{\partial r^2}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\right. \\
&\qquad -2i\frac{mgt}{\hbar}\frac{\partial\Psi_0}{\partial r}\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right] \\
&\qquad \left.-\frac{m^2g^2t^2}{\hbar^2}\Psi_0(r,t)\exp\left[-i\frac{mgt}{\hbar}\left(r-\frac{1}{3}gt^2\right)\right]\right\}dr \\
&= -\hbar^2\left[\int_{-\infty}^{\infty}\Psi_0^*(r,t)\frac{\partial^2\Psi_0}{\partial r^2}\,dr-2i\frac{mgt}{\hbar}\underbrace{\int_{-\infty}^{\infty}\Psi_0^*(r,t)\frac{\partial\Psi_0}{\partial r}\,dr}_{=0}-\frac{m^2g^2t^2}{\hbar^2}\underbrace{\int_{-\infty}^{\infty}\Psi_0^*(r,t)\Psi_0(r,t)\,dr}_{=1}\right] \\
&= -\hbar^2\int_{-\infty}^{\infty}\Psi_0^*(r,t)\frac{\partial^2\Psi_0}{\partial r^2}\,dr+m^2g^2t^2 \\
&= -\hbar^2\int_{-\infty}^{\infty}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1-\frac{2i\hbar at}{m}}}\exp\left(-\frac{ar^2}{1-\frac{2i\hbar at}{m}}\right)\right\}\frac{\partial^2}{\partial r^2}\left\{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ar^2}{1+\frac{2i\hbar at}{m}}\right)\right\}dr \\
&\qquad +m^2g^2t^2 \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{ar^2}{1-\frac{2i\hbar at}{m}}\right)\left[\frac{2a}{1+\frac{2i\hbar at}{m}}\exp\left(-\frac{ar^2}{1+\frac{2i\hbar at}{m}}\right)\left(\frac{2a}{1+\frac{2i\hbar at}{m}}r^2-1\right)\right]dr \\
&\qquad +m^2g^2t^2 \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\frac{2a}{1+\frac{2i\hbar at}{m}}\left[\frac{2a}{1+\frac{2i\hbar at}{m}}\int_{-\infty}^{\infty}r^2\exp\left(-\frac{2ar^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dr-\int_{-\infty}^{\infty}\exp\left(-\frac{2ar^2}{1+\frac{4\hbar^2a^2t^2}{m^2}}\right)dr\right] \\
&\qquad +m^2g^2t^2 \\
&= -\hbar^2\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\frac{4\hbar^2a^2t^2}{m^2}}}\frac{2a}{1+\frac{2i\hbar at}{m}}\left[2\frac{2a}{1+\frac{2i\hbar at}{m}}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}}{2}\right)^3-2\cdot\sqrt{\pi}\left(\frac{\sqrt{\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}}}{2}\right)\right]+m^2g^2t^2 \\
&= -\hbar^2\frac{2a}{1+\frac{2i\hbar at}{m}}\left(\frac{a}{1+\frac{2i\hbar at}{m}}\cdot\frac{1+\frac{4\hbar^2a^2t^2}{m^2}}{2a}-1\right)+m^2g^2t^2 \\
&= a\hbar^2+m^2g^2t^2
\end{aligned}$$

The standard deviation in p is then

$$\langle p^2 \rangle = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{(a\hbar^2 + m^2 g^2 t^2) - m^2 g^2 t^2} = \hbar\sqrt{a}.$$

As a result, the uncertainty product is

$$\sigma_x \sigma_p = \sqrt{\frac{1}{4a} + \frac{\hbar^2 a t^2}{m^2}}(\hbar\sqrt{a}) = \frac{\hbar}{2}\sqrt{1 + \frac{4\hbar^2 a^2 t^2}{m^2}},$$

which is consistent with Heisenberg's principle ($\sigma_x \sigma_p \geq \hbar/2$). The product comes closest to the limit at $t = 0$.

Part (d)

Here the final result for $\Psi(x,t)$ stated in the beginning will be derived. The aim is to solve the Schrödinger equation with a gravitational potential for the wave function that is initially a gaussian wave packet centered at $x = 0$.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + mgx\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

$$\Psi(x,0) = Ae^{-ax^2}$$

Start by normalizing the initial wave function.

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2\, dx = A^2 \int_{-\infty}^{\infty} e^{-2ax^2}\, dx = A^2\sqrt{\frac{\pi}{2a}} \quad \rightarrow \quad A = \left(\frac{2a}{\pi}\right)^{1/4}$$

Since the PDE is linear and defined over $-\infty < x < \infty$, the Fourier transform can be used to solve it. The Fourier transform of a function is defined here as

$$\mathcal{F}\{\Psi(x,t)\} = \hat{\Psi}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\Psi(x,t)\, dx.$$

As a result, the derivatives of Ψ transform as follows.

$$\mathcal{F}\left\{\frac{\partial \Psi}{\partial t}\right\} = \frac{\partial \hat{\Psi}}{\partial t}$$

$$\mathcal{F}\left\{\frac{\partial^2 \Psi}{\partial x^2}\right\} = (ik)^2 \hat{\Psi}(k,t)$$

Also,

$$\begin{aligned}\mathcal{F}\{x\Psi(x,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} x\Psi(x,t)\, dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{-i}\frac{\partial}{\partial k} e^{-ikx}\Psi(x,t)\, dx \\ &= i\frac{\partial}{\partial k}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\Psi(x,t)\, dx\right] \\ &= i\frac{\partial \hat{\Psi}}{\partial k}.\end{aligned}$$

Take the Fourier transform of both sides of Schrödinger's equation and the initial condition.

$$\mathcal{F}\left\{i\hbar\frac{\partial\Psi}{\partial t}\right\} = \mathcal{F}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mgx\Psi(x,t)\right\} \qquad \mathcal{F}\{\Psi(x,0)\} = \mathcal{F}\{Ae^{-ax^2}\} = \frac{A}{\sqrt{2a}}e^{-k^2/(4a)}$$

Use the fact that the transform is a linear operator.

$$i\hbar\mathcal{F}\left\{\frac{\partial\Psi}{\partial t}\right\} = -\frac{\hbar^2}{2m}\mathcal{F}\left\{\frac{\partial^2\Psi}{\partial x^2}\right\} + mg\mathcal{F}\left\{x\Psi(x,t)\right\}$$

Substitute each of the formulas.

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t} = -\frac{\hbar^2}{2m}(ik)^2\hat{\Psi}(k,t) + img\frac{\partial\hat{\Psi}}{\partial k}$$

Divide both sides by $i\hbar$ and bring both derivatives to the left side.

$$\frac{\partial\hat{\Psi}}{\partial t} - \frac{mg}{\hbar}\frac{\partial\hat{\Psi}}{\partial k} = -\frac{i\hbar k^2}{2m}\hat{\Psi}(k,t) \tag{2}$$

This is a first-order PDE for $\hat{\Psi}(k,t)$, so apply the method of characteristics to solve it. Recall that the differential of $\hat{\Psi}$ is

$$d\hat{\Psi} = \frac{\partial\hat{\Psi}}{\partial t}\,dt + \frac{\partial\hat{\Psi}}{\partial k}\,dk.$$

Dividing both sides by dt results in the fundamental relationship between the total derivative of $\hat{\Psi}$ with respect to time and its partial derivatives.

$$\frac{d\hat{\Psi}}{dt} = \frac{\partial\hat{\Psi}}{\partial t} + \frac{dk}{dt}\frac{\partial\hat{\Psi}}{\partial k}$$

Along the characteristic curves in the tk-plane defined by

$$\frac{dk}{dt} = -\frac{mg}{\hbar}, \quad k(\xi,0) = \xi, \tag{3}$$

where ξ is a characteristic coordinate, equation (2) reduces to an ODE.

$$\frac{d\hat{\Psi}}{dt} = -\frac{i\hbar k^2}{2m}\hat{\Psi}(k,t) \tag{4}$$

Solve equation (3) by integrating both sides with respect to t.

$$k = -\frac{mg}{\hbar}t + \xi \quad \rightarrow \quad \xi = k + \frac{mg}{\hbar}t$$

Substitute this formula for k into equation (4) and then solve the ODE.

$$\frac{d\hat{\Psi}}{dt} = -\frac{i\hbar}{2m}\left(-\frac{mg}{\hbar}t + \xi\right)^2\hat{\Psi}(\xi,t)$$

$$\frac{\frac{d\hat{\Psi}}{dt}}{\hat{\Psi}} = -\frac{i\hbar}{2m}\left(-\frac{mg}{\hbar}t + \xi\right)^2$$

$$\frac{d}{dt}\ln\hat{\Psi} = -\frac{img^2}{2\hbar}t^2 + ig\xi t - \frac{i\hbar}{2m}\xi^2$$

Integrate both sides with respect to t.

$$\ln \hat{\Psi} = -\frac{img^2}{6\hbar}t^3 + \frac{ig}{2}\xi t^2 - \frac{i\hbar}{2m}\xi^2 t + f(\xi)$$

Here f is an arbitrary function that is constant along any of the characteristic curves in the family. Exponentiate both sides, using a new arbitrary function $F(\xi)$ for $e^{f(\xi)}$.

$$\begin{aligned}\hat{\Psi}(\xi, t) &= \exp\left[-\frac{img^2}{6\hbar}t^3 + \frac{ig}{2}\xi t^2 - \frac{i\hbar}{2m}\xi^2 t + f(\xi)\right] \\ &= F(\xi)\exp\left(-\frac{img^2}{6\hbar}t^3\right)\exp\left(\frac{ig}{2}\xi t^2 - \frac{i\hbar}{2m}\xi^2 t\right)\end{aligned}$$

Now that $\hat{\Psi}$ is known, write ξ in terms of k and t.

$$\begin{aligned}\hat{\Psi}(k, t) &= F\left(k + \frac{mg}{\hbar}t\right)\exp\left(-\frac{img^2}{6\hbar}t^3\right)\exp\left[\frac{ig}{2}\left(k + \frac{mg}{\hbar}t\right)t^2 - \frac{i\hbar}{2m}\left(k + \frac{mg}{\hbar}t\right)^2 t\right] \\ &= F\left(k + \frac{mg}{\hbar}t\right)\exp\left(-\frac{img^2}{6\hbar}t^3\right)\exp\left(-\frac{i\hbar t}{2m}k^2 - \frac{igt^2}{2}k\right)\end{aligned}$$

In order to determine F, set $t = 0$ in this formula and apply the transformed initial condition.

$$\hat{\Psi}(k, 0) = F(k) = \frac{A}{\sqrt{2a}}e^{-k^2/(4a)} = \frac{1}{(2\pi a)^{1/4}}\exp\left(-\frac{k^2}{4a}\right)$$

As a result,

$$F\left(k + \frac{mg}{\hbar}t\right) = \frac{1}{(2\pi a)^{1/4}}\exp\left[-\frac{1}{4a}\left(k + \frac{mg}{\hbar}t\right)^2\right] = \frac{1}{(2\pi a)^{1/4}}\exp\left(-\frac{m^2g^2}{4\hbar^2 a}t^2\right)\exp\left(-\frac{1}{4a}k^2 - \frac{mgt}{2\hbar a}k\right),$$

which means the transformed wave function is

$$\begin{aligned}\hat{\Psi}(k, t) &= \frac{1}{(2\pi a)^{1/4}}\exp\left(-\frac{m^2g^2}{4\hbar^2 a}t^2\right)\exp\left(-\frac{1}{4a}k^2 - \frac{mgt}{2\hbar a}k\right)\exp\left(-\frac{img^2}{6\hbar}t^3\right)\exp\left(-\frac{i\hbar t}{2m}k^2 - \frac{igt^2}{2}k\right) \\ &= \frac{1}{(2\pi a)^{1/4}}\exp\left(-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2\right)\exp\left[\left(-\frac{1}{4a} - \frac{i\hbar t}{2m}\right)k^2 + \left(-\frac{mgt}{2\hbar a} - \frac{igt^2}{2}\right)k\right].\end{aligned}$$

Now take the inverse Fourier transform of $\hat{\Psi}(k, t)$ to get $\Psi(x, t)$, the desired solution.

$$\begin{aligned}\Psi(x, t) &= \mathcal{F}^{-1}\left\{\hat{\Psi}(k, t)\right\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\frac{1}{(2\pi a)^{1/4}}\exp\left(-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2\right)\exp\left[\left(-\frac{1}{4a} - \frac{i\hbar t}{2m}\right)k^2 + \left(-\frac{mgt}{2\hbar a} - \frac{igt^2}{2}\right)k\right]dk \\ &= \frac{1}{(8\pi^3 a)^{1/4}}\exp\left(-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2\right)\int_{-\infty}^{\infty}\exp\left[-\frac{m + 2i\hbar at}{4ma}k^2 + \left(-\frac{mgt}{2\hbar a} - \frac{igt^2}{2} + ix\right)k\right]dk \\ &= \frac{1}{(8\pi^3 a)^{1/4}}\exp\left(-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2\right)\int_{-\infty}^{\infty}\exp\left\{-\frac{m + 2i\hbar at}{4ma}\left[k^2 + \frac{2m}{\hbar}\frac{gt(m + i\hbar at) - 2i\hbar ax}{m + 2i\hbar at}k\right]\right\}dk\end{aligned}$$

Complete the square in the exponent and evaluate the integral.

$$\Psi(x,t) = \frac{1}{(8\pi^3 a)^{1/4}} \exp\left(-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2\right) \int_{-\infty}^{\infty} \exp\left\{-\frac{m+2i\hbar at}{4ma}\left[k + \frac{m}{\hbar}\frac{gt(m+i\hbar at) - 2i\hbar ax}{m+2i\hbar at}\right]^2\right\} \times \exp\left\{\frac{m}{4\hbar^2 a}\frac{[gt(m+i\hbar at)-2i\hbar ax]^2}{m+2i\hbar at}\right\} dk$$

$$= \frac{1}{(8\pi^3 a)^{1/4}} \exp\left\{-\frac{img^2}{6\hbar}t^3 - \frac{m^2g^2}{4\hbar^2 a}t^2 + \frac{m}{4\hbar^2 a}\frac{[gt(m+i\hbar at)-2i\hbar ax]^2}{m+2i\hbar at}\right\} \int_{-\infty}^{\infty} \exp\left(-\frac{m+2i\hbar at}{4ma}u^2\right) du$$

$$= \frac{2}{(8\pi^3 a)^{1/4}} \exp\left(\frac{\frac{ag^2t^4}{12} - \frac{img^2}{6\hbar}t^3 + agxt^2 - \frac{imgx}{\hbar}t - ax^2}{1+\frac{2i\hbar at}{m}}\right) \int_0^{\infty} \exp\left[-\frac{u^2}{\left(\sqrt{\frac{4ma}{m+2i\hbar at}}\right)^2}\right] du$$

$$= \frac{2}{(8\pi^3 a)^{1/4}} \exp\left[\frac{\left(-ax^2 - agxt^2 - \frac{ag^2t^4}{4}\right) + \frac{ag^2t^4}{3} - \frac{img^2}{6\hbar}t^3 + 2agxt^2 - \frac{imgx}{\hbar}t}{1+\frac{2i\hbar at}{m}}\right] \cdot \sqrt{\pi}\left(\frac{\sqrt{\frac{4ma}{m+2i\hbar at}}}{2}\right)$$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \exp\left[\frac{\left(-ax^2 - agxt^2 - \frac{ag^2t^4}{4}\right) + \frac{ag^2t^4}{3} - \frac{img^2}{6\hbar}t^3 + 2agxt^2 - \frac{imgx}{\hbar}t}{1+\frac{2i\hbar at}{m}}\right] \sqrt{\frac{m}{m+2i\hbar at}}$$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \exp\left[\frac{\left(-ax^2 - agxt^2 - \frac{ag^2t^4}{4}\right)}{1+\frac{2i\hbar at}{m}}\right] \exp\left(\frac{\frac{ag^2t^4}{3} - \frac{img^2}{6\hbar}t^3 + 2agxt^2 - \frac{imgx}{\hbar}t}{1+\frac{2i\hbar at}{m}}\right) \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}}$$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left[\frac{-a\left(x+\frac{1}{2}gt^2\right)^2}{1+\frac{2i\hbar at}{m}}\right] \exp\left[\frac{-\frac{imgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\left(1+\frac{2i\hbar at}{m}\right)}{1+\frac{2i\hbar at}{m}}\right]$$

Therefore,

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left[-\frac{a\left(x+\frac{1}{2}gt^2\right)^2}{1+\frac{2i\hbar at}{m}}\right] \exp\left[-i\frac{mgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right].$$