Problem 2.52

Consider the potential

$$V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax),$$

where a is a positive constant, and "sech" stands for hyperbolic secant.

(a) Graph this potential.

(b) Check that this potential has the ground state

$$\psi_0(x) = A\operatorname{sech}(ax),$$

and find its energy. Normalize ψ_0, and sketch its graph.

(c) Show that the function

$$\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik + a}\right)e^{ikx},$$

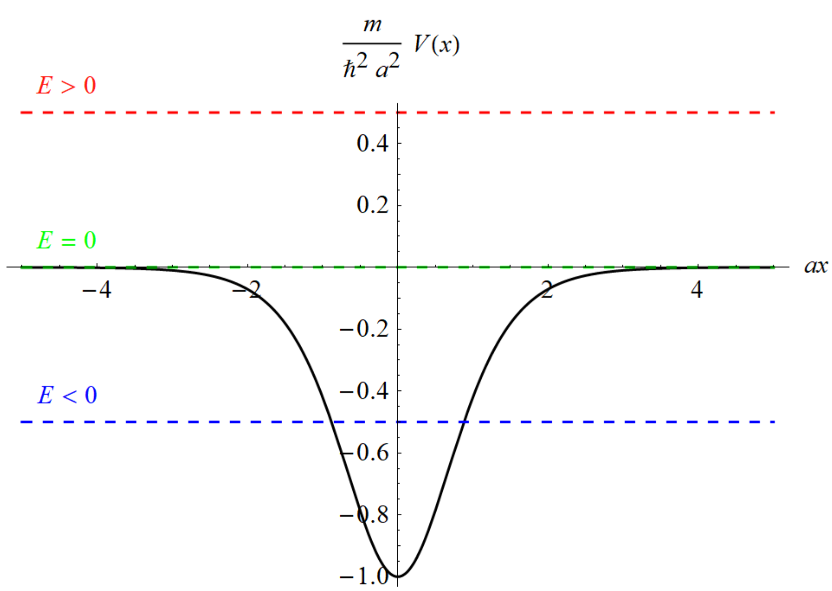
(where $k \equiv \sqrt{2mE}/\hbar$, as usual) solves the Schrödinger equation for any (positive) energy E. Since $\tanh z \to -1$ as $z \to -\infty$,

$$\psi_k(x) \approx Ae^{ikx}, \quad \text{for large negative } x.$$

This represents, then, a wave coming in from the left with *no accompanying reflected wave* (i.e. no term $\exp(-ikx)$). What is the asymptotic form of $\psi_k(x)$ at large *positive* x? What are R and T, for this potential? *Comment:* This is a famous example of a **reflectionless potential**—every incident particle, regardless its energy, passes right through.[62]

Solution

Part (a)



[62] R. E. Crandall and B. R. Litt, *Annals of Physics*, **146**, 458 (1983).

Part (b)

The goal here is merely to verify that $\psi_0(x) = A\operatorname{sech}(ax)$ satisfies the time-independent Schrödinger equation (TISE),

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\psi(x) = E\psi(x),$$

and to determine the corresponding energy E. Evaluate the left side with $\psi = \psi_0(x)$.

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\psi_0(x) &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}[A\operatorname{sech}(ax)] - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)[A\operatorname{sech}(ax)] \\
&= -\frac{\hbar^2}{2m}\frac{d}{dx}[-Aa\operatorname{sech}(ax)\tanh(ax)] - A\frac{\hbar^2 a^2}{m}\operatorname{sech}^3(ax) \\
&= -\frac{\hbar^2}{2m}A[-a^2\operatorname{sech}^3(ax) + a^2\operatorname{sech}(ax)\tanh^2(ax)] - A\frac{\hbar^2 a^2}{m}\operatorname{sech}^3(ax) \\
&= A\frac{\hbar^2 a^2}{2m}\operatorname{sech}^3(ax) - A\frac{\hbar^2 a^2}{2m}\operatorname{sech}(ax)\tanh^2(ax) - A\frac{\hbar^2 a^2}{m}\operatorname{sech}^3(ax) \\
&= -A\frac{\hbar^2 a^2}{2m}\operatorname{sech}^3(ax) - A\frac{\hbar^2 a^2}{2m}\operatorname{sech}(ax)\tanh^2(ax) \\
&= -A\frac{\hbar^2 a^2}{2m}\operatorname{sech}^3(ax) - A\frac{\hbar^2 a^2}{2m}\operatorname{sech}(ax)[1 - \operatorname{sech}^2(ax)] \\
&= \cancel{-A\frac{\hbar^2 a^2}{2m}\operatorname{sech}^3(ax)} - A\frac{\hbar^2 a^2}{2m}\operatorname{sech}(ax) + \cancel{A\frac{\hbar^2 a^2}{2m}\operatorname{sech}^3(ax)} \\
&= -A\frac{\hbar^2 a^2}{2m}\operatorname{sech}(ax) \\
&= -\frac{\hbar^2 a^2}{2m}[A\operatorname{sech}(ax)] \\
&= -\frac{\hbar^2 a^2}{2m}\psi_0(x)
\end{aligned}$$

Because the left side yields a constant times $\psi_0(x)$, $\psi_0(x)$ is a solution to the TISE. The constant in front is the energy associated with $\psi_0(x)$.

$$E_0 = -\frac{\hbar^2 a^2}{2m}$$

Normalize $\psi_0(x)$ now.

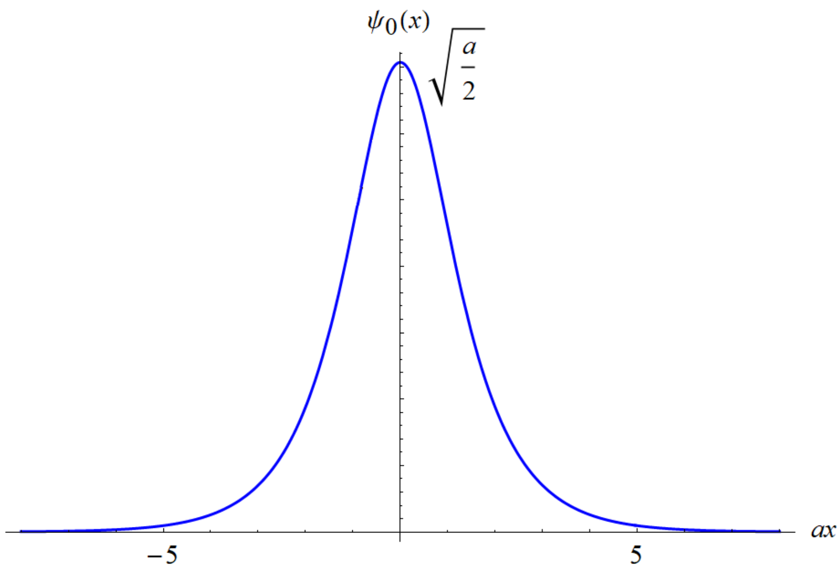
$$1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2\,dx = \int_{-\infty}^{\infty} A^2\operatorname{sech}^2(ax)\,dx = A^2\int_{-\infty}^{\infty}\operatorname{sech}^2 z\,\frac{dz}{a} = \frac{A^2}{a}(\tanh z)\Big|_{-\infty}^{\infty} = \frac{A^2}{a}(2)$$

Solve for A.

$$A = \sqrt{\frac{a}{2}}$$

Therefore,

$$\psi_0(x) = \sqrt{\frac{a}{2}}\,\text{sech}(ax).$$



Part (c)

The goal here is merely to verify that

$$\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} = A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}$$

satisfies the time-independent Schrödinger equation (TISE),

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\psi(x) = E\psi(x).$$

Evaluate the left side with $\psi = \psi_k(x)$.

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\psi_k(x) &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left[A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}\right] \\
&\quad - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\left[A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}\right] \\
&= -\frac{\hbar^2}{2m}\left[A\left(\frac{ik}{ik+a}\right)\frac{d^2}{dx^2}e^{ikx} - A\left(\frac{a}{ik+a}\right)\frac{d^2}{dx^2}\tanh(ax)e^{ikx}\right] \\
&\quad - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\left[A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}\right]
\end{aligned}$$

Continue the simplification.

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\psi_k(x) &= -\frac{\hbar^2}{2m}\left\{A\left(\frac{ik}{ik+a}\right)(ik)^2 e^{ikx}\right. \\
&\qquad \left. -A\left(\frac{a}{ik+a}\right)\frac{d}{dx}[a\operatorname{sech}^2(ax)e^{ikx} + ik\tanh(ax)e^{ikx}]\right\} \\
&\qquad - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\left[A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}\right] \\
&= -\frac{\hbar^2}{2m}\left\{-A\left(\frac{ik}{ik+a}\right)k^2 e^{ikx} - A\left(\frac{a}{ik+a}\right)[-2a^2\operatorname{sech}^2(ax)\tanh(ax)e^{ikx}\right. \\
&\qquad + 2ika\operatorname{sech}^2(ax)e^{ikx} \\
&\qquad \left. - k^2\tanh(ax)e^{ikx}]\right\} \\
&\qquad - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\left[A\left(\frac{ik}{ik+a}\right)e^{ikx} - A\left(\frac{a}{ik+a}\right)\tanh(ax)e^{ikx}\right] \\
&= A\frac{\hbar^2}{2m(ik+a)}\left[ik^3 e^{ikx} - \cancel{2a^3\operatorname{sech}^2(ax)\tanh(ax)e^{ikx}}\right. \\
&\qquad \left. + \cancel{2ika^2\operatorname{sech}^2(ax)e^{ikx}} - ak^2\tanh(ax)e^{ikx}\right] \\
&\qquad - A\frac{\hbar^2}{2m(ik+a)}\left[\cancel{2ika^2\operatorname{sech}^2(ax)e^{ikx}} - \cancel{2a^3\operatorname{sech}^2(ax)\tanh(ax)e^{ikx}}\right] \\
&= A\frac{\hbar^2}{2m(ik+a)}k^2[ik - a\tanh(ax)]e^{ikx} \\
&= \frac{\hbar^2 k^2}{2m}\left[A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx}\right] \\
&= \frac{\hbar^2 k^2}{2m}\psi_k(x)
\end{aligned}
$$

Because the left side yields a constant times $\psi_k(x)$, $\psi_k(x)$ is a solution to the TISE. The constant in front is the energy associated with $\psi_k(x)$:

$$\frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2mE}{\hbar^2}\right) = E.$$

Since $\tanh(ax) \to 1$ as $x \to \infty$,

$$\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} \approx A\left(\frac{ik-a}{ik+a}\right)e^{ikx}$$

for large positive x. Like before, this is a plane wave going from left to right with no corresponding reflected wave. The reflection and transmission coefficients are defined by

$$R = \left|\frac{\text{reflected probability current}}{\text{incident probability current}}\right| \qquad T = \left|\frac{\text{transmitted probability current}}{\text{incident probability current}}\right|,$$

where the probability current is [noting that $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$]

$$\begin{aligned} J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{\partial}{\partial x}[\psi^*(x)e^{iEt/\hbar}] - [\psi^*(x)e^{iEt/\hbar}]\frac{\partial}{\partial x}[\psi(x)e^{-iEt/\hbar}]\right\} \\ &= \frac{i\hbar}{2m}\left\{[\psi(x)e^{-iEt/\hbar}]\frac{d\psi^*}{dx}e^{iEt/\hbar} - [\psi^*(x)e^{iEt/\hbar}]\frac{d\psi}{dx}e^{-iEt/\hbar}\right\} \\ &= \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} R &= \left|\frac{\text{reflected probability current}}{\text{incident probability current}}\right| \\ &= \left|\frac{0}{\frac{i\hbar}{2m}\left\{\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\frac{d}{dx}\left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right] - \left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right]\frac{d}{dx}\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\right\}}\right| \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} T &= \left|\frac{\text{transmitted probability current}}{\text{incident probability current}}\right| \\ &= \left|\frac{\frac{i\hbar}{2m}\left\{\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\frac{d}{dx}\left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right] - \left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right]\frac{d}{dx}\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\right\}}{\frac{i\hbar}{2m}\left\{\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\frac{d}{dx}\left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right] - \left[A\left(\frac{-ik-a}{-ik+a}\right)e^{-ikx}\right]\frac{d}{dx}\left[A\left(\frac{ik-a}{ik+a}\right)e^{ikx}\right]\right\}}\right| \\ &= 1. \end{aligned}$$

Part (d)

The Schrödinger equation will be solved here on the whole line with $V(x) = -(\hbar^2 a^2/m)\,\text{sech}^2(ax)$

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

and the usual Dirichlet boundary conditions, $\Psi \to 0$ as $x \to \pm\infty$. Because the Schrödinger equation is linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)[\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax)$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax) = E$$

As a result of applying the method of separation of variables, Schrödinger's equation has reduced to two ODEs—one in each of the independent variables, x and t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\hbar^2 a^2}{m}\,\text{sech}^2(ax) &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions of this second equation are satisfied are known as eigenvalues (or eigenenergies in this context), and the nontrivial solutions $\psi(x)$ that satisfy this second equation are known as eigenfunctions (or eigenstates in this context). Multiply both sides of the TISE by $(-2m/\hbar^2)\psi(x)$.

$$\frac{d^2\psi}{dx^2} + 2a^2\,\text{sech}^2(ax)\psi(x) = -\frac{2mE}{\hbar^2}\psi(x) \tag{1}$$

Make the substitution, $z = ax$. Use the chain rule to write the second derivative of ψ in terms of this new variable.

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{d\psi}{dz}\frac{dz}{dx} = a\frac{d\psi}{dz} \\ \frac{d^2\psi}{dx^2} &= \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left(a\frac{d\psi}{dz}\right) = a^2\frac{d^2\psi}{dz^2} \end{aligned}$$

Consequently,

$$a^2\frac{d^2\psi}{dz^2} + 2a^2(\operatorname{sech}^2 z)\psi(z) = -\frac{2mE}{\hbar^2}\psi(z).$$

Divide both sides by a^2, let $\mathcal{E}$ represent $2mE/(\hbar^2 a^2)$, and bring all terms to the left side.

$$\frac{d^2\psi}{dz^2} + (\mathcal{E} + 2\operatorname{sech}^2 z)\psi(z) = 0 \tag{2}$$

Now make the substitution, $u = \operatorname{sech}^2 z$. Use the chain rule again to write this equation in terms of u.

$$\begin{aligned}
\frac{d\psi}{dz} &= \frac{d\psi}{du}\frac{du}{dz} = \frac{d\psi}{du}(-2\operatorname{sech}^2 z\tanh z) \\
\frac{d^2\psi}{dz^2} &= \frac{d}{dz}\left(\frac{d\psi}{dz}\right) = \frac{d}{dz}\left[\frac{d\psi}{du}(-2\operatorname{sech}^2 z\tanh z)\right] \\
&= \frac{d}{dz}\left(\frac{d\psi}{du}\right)(-2\operatorname{sech}^2 z\tanh z) + \frac{d\psi}{du}\frac{d}{dz}(-2\operatorname{sech}^2 z\tanh z) \\
&= \frac{du}{dz}\frac{d}{du}\left(\frac{d\psi}{du}\right)(-2\operatorname{sech}^2 z\tanh z) + \frac{d\psi}{du}(4\operatorname{sech}^2 z\tanh^2 z - 2\operatorname{sech}^4 z) \\
&= \frac{d^2\psi}{du^2}(-2\operatorname{sech}^2 z\tanh z)^2 + \frac{d\psi}{du}(4\operatorname{sech}^2 z\tanh^2 z - 2\operatorname{sech}^4 z) \\
&= \frac{d^2\psi}{du^2}(4\operatorname{sech}^4 z\tanh^2 z) + \frac{d\psi}{du}(4\operatorname{sech}^2 z\tanh^2 z - 2\operatorname{sech}^4 z) \\
&= \frac{d^2\psi}{du^2}[4\operatorname{sech}^4 z(1-\operatorname{sech}^2 z)] + \frac{d\psi}{du}[4\operatorname{sech}^2 z(1-\operatorname{sech}^2 z) - 2\operatorname{sech}^4 z] \\
&= \frac{d^2\psi}{du^2}[4u^2(1-u)] + \frac{d\psi}{du}[4u(1-u) - 2u^2] \\
&= 4u^2(1-u)\frac{d^2\psi}{du^2} + 2u(2-3u)\frac{d\psi}{du}
\end{aligned}$$

As a result, equation (2) becomes

$$4u^2(1-u)\frac{d^2\psi}{du^2} + 2u(2-3u)\frac{d\psi}{du} + (\mathcal{E} + 2u)\psi(u) = 0.$$

Check to see if zero is an eigenvalue: $\mathcal{E} = 0$.

$$4u^2(1-u)\frac{d^2\psi}{du^2} + 2u(2-3u)\frac{d\psi}{du} + 2u\psi(u) = 0$$

Divide both sides by $4u$.

$$u(1-u)\frac{d^2\psi}{du^2} + \left(1 - \frac{3}{2}u\right)\frac{d\psi}{du} + \frac{1}{2}\psi(u) = 0 \tag{3}$$

This is the hypergeometric equation with $a = -1/2$ and $b = 1$ and $c = 1$, so one of its solutions is

$$\psi(u) = {}_2F_1\left(-\frac{1}{2}, 1; 1; u\right) = \sqrt{1-u}.$$

Just for reference, the hypergeometric equation is

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0,$$

and one of its two linearly independent solutions is denoted by

$$y(x) = {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}.$$

Use the method of reduction of order to obtain the general solution: Substitute $\psi(u) = \sqrt{1-u}\,w(u)$ into equation (3).

$$u(1-u)\frac{d^2}{du^2}[\sqrt{1-u}\,w(u)] + \left(1-\frac{3}{2}u\right)\frac{d}{du}[\sqrt{1-u}\,w(u)] + \frac{1}{2}[\sqrt{1-u}\,w(u)] = 0$$

Simplify the left side.

$$-\frac{1}{2}\sqrt{1-u}\left[2u(u-1)\frac{d^2w}{du^2} + (5u-2)\frac{dw}{du}\right] = 0$$

This is a first-order linear ODE for dw/du, which can be solved with an integrating factor.

$$w(u) = C_1 + C_2\left[\frac{1}{\sqrt{1-u}} - \tanh^{-1}\left(\sqrt{1-u}\right)\right]$$

Consequently,

$$\psi(u) = C_1\sqrt{1-u} + C_2\left[1-\sqrt{1-u}\tanh^{-1}\left(\sqrt{1-u}\right)\right].$$

Changing back to the original variables yields

$$\begin{aligned}\psi(x) &= C_1\sqrt{1-\operatorname{sech}^2(ax)} + C_2\left[1-\sqrt{1-\operatorname{sech}^2(ax)}\tanh^{-1}\left(\sqrt{1-\operatorname{sech}^2(ax)}\right)\right]\\ &= C_1\sqrt{\tanh^2(ax)} + C_2\left[1-\sqrt{\tanh^2(ax)}\tanh^{-1}\left(\sqrt{\tanh^2(ax)}\right)\right]\\ &= D_1\tanh(ax) + D_2\left[1-ax\tanh(ax)\right].\end{aligned}$$

In order to satisfy the boundary conditions, $\psi \to 0$ as $x \to \pm\infty$, D_1 and D_2 must be set to zero. This leads to the trivial solution, $\psi(x) = 0$, so zero is not an eigenvalue. Now check to see if there are negative eigenvalues: $\mathcal{E} = -\gamma^2$.

$$4u^2(1-u)\frac{d^2\psi}{du^2} + 2u(2-3u)\frac{d\psi}{du} + (-\gamma^2 + 2u)\psi(u) = 0$$

As it is, this equation can't be solved in terms of hypergeometric functions like before because $-\gamma^2$ is not multiplied by u. Make the substitution, $\psi(u) = u^r f(u)$, then. This is motivated from the u^2 factor in front of the second derivative and the u factor in front of the first derivative.

$$\begin{aligned}\frac{d\psi}{du} &= ru^{r-1}f(u) + u^r\frac{df}{du}\\ \frac{d^2\psi}{du^2} &= r(r-1)u^{r-2}f(u) + 2ru^{r-1}\frac{df}{du} + u^r\frac{d^2f}{du^2}\end{aligned}$$

Plug these formulas into the ODE.

$$4u^2(1-u)\left[r(r-1)u^{r-2}f(u) + 2ru^{r-1}\frac{df}{du} + u^r\frac{d^2f}{du^2}\right] + 2u(2-3u)\left[ru^{r-1}f(u) + u^r\frac{df}{du}\right] + (-\gamma^2+2u)u^rf(u) = 0$$

Divide both sides by u^r and simplify the left side.

$$4u^2(1-u)\frac{d^2f}{du^2} + (4u + 8ru - 6u^2 - 8ru^2)\frac{df}{du} + (4r^2 - \gamma^2 + 2u - 2ru - 4r^2u)f(u) = 0$$

Choose r so that

$$4r^2 - \gamma^2 = 0 \quad \rightarrow \quad r = \frac{\gamma}{2}.$$

As a result,

$$4u^2(1-u)\frac{d^2f}{du^2} + (4u + 4\gamma u - 6u^2 - 4\gamma u^2)\frac{df}{du} + (2u - \gamma u - \gamma^2 u)f(u) = 0.$$

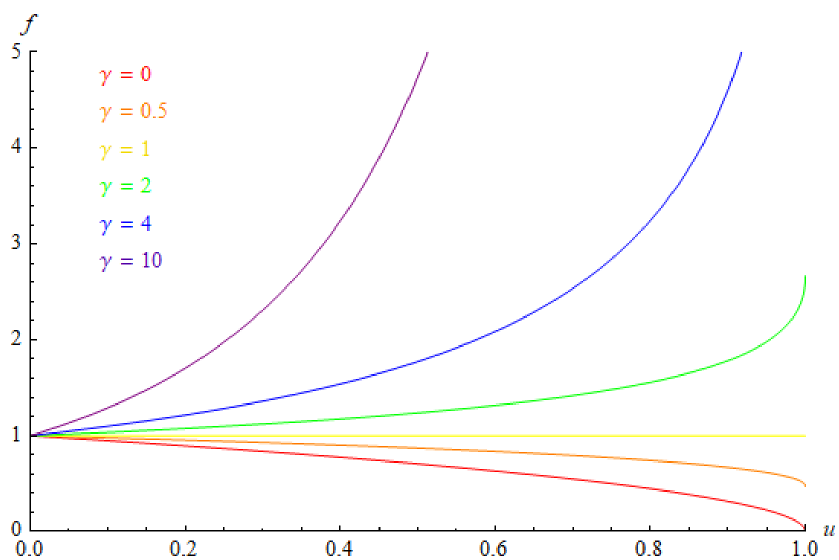
Divide both sides by $4u$.

$$u(1-u)\frac{d^2f}{du^2} + \left(1 + \gamma - \frac{3}{2}u - \gamma u\right)\frac{df}{du} + \left(\frac{1}{2} - \frac{\gamma}{4} - \frac{\gamma^2}{4}\right)f(u) = 0 \tag{4}$$

This ODE for f is a hypergeometric equation with $a = -1/2 + \gamma/2$, $b = 1 + \gamma/2$, and $c = 1 + \gamma$. That means one solution is

$$f(u) = {}_2F_1\left(-\frac{1}{2} + \frac{\gamma}{2}, 1 + \frac{\gamma}{2}; 1 + \gamma; u\right).$$

Use the wag-the-dog method to determine appropriate values of γ: Plot $f(u)$ versus u for several values of γ and see where the graph flips over.



The graph only flips over once at $\gamma = 1$. Note that negative values of γ are not considered because $u = \operatorname{sech}^2 z$, and $\psi(u) = u^{\gamma/2}f(u)$ would blow up as a result. With $\gamma = 1$, equation (4) becomes

$$u(1-u)\frac{d^2f}{du^2} + \left(2 - \frac{5}{2}u\right)\frac{df}{du} = 0,$$

which is a first-order linear ODE for df/du, solvable with an integrating factor. The general solution is

$$f(u) = C_3 + C_4\left[\frac{\sqrt{1-u}}{u} + \tanh^{-1}\left(\sqrt{1-u}\right)\right],$$

which means

$$\psi(u) = u^{1/2} f(u) = C_3\sqrt{u} + C_4\left[\sqrt{\frac{1-u}{u}} + \sqrt{u}\tanh^{-1}\left(\sqrt{1-u}\right)\right].$$

Changing back to the original variables yields

$$\begin{aligned}
\psi(x) &= C_3\sqrt{\operatorname{sech}^2(ax)} + C_4\left[\sqrt{\frac{1-\operatorname{sech}^2(ax)}{\operatorname{sech}^2(ax)}} + \sqrt{\operatorname{sech}^2(ax)}\tanh^{-1}\left(\sqrt{1-\operatorname{sech}^2(ax)}\right)\right] \\
&= C_3\sqrt{\operatorname{sech}^2(ax)} + C_4\left[\sqrt{\frac{\tanh^2(ax)}{\operatorname{sech}^2(ax)}} + \sqrt{\operatorname{sech}^2(ax)}\tanh^{-1}\left(\sqrt{\tanh^2(ax)}\right)\right] \\
&= D_3\operatorname{sech}(ax) + D_4[\sinh(ax) + ax\operatorname{sech}(ax)].
\end{aligned}$$

In order to satisfy the boundary conditions, $\psi \to 0$ as $x \to \pm\infty$, D_4 must be set to zero. Normalization determines D_3 as shown in part (b).

$$\psi(x) = \sqrt{\frac{a}{2}}\operatorname{sech}(ax)$$

Remarkably, this is the only bound state. The corresponding energy is

$$\mathcal{E} = \frac{2mE}{\hbar^2 a^2} = -1 \quad \to \quad E = -\frac{\hbar^2 a^2}{2m}.$$

Solving the ODE in t yields $\phi(t) = e^{-iEt/\hbar}$. Therefore, one solution to Schrödinger's equation is

$$\boxed{\Psi(x,t) = \phi(x)\phi(t) = \sqrt{\frac{a}{2}}\operatorname{sech}(ax)\exp\left(i\frac{\hbar a^2}{2m}t\right).}$$

Return to equation (1), now assuming that E is positive.

$$\frac{d^2\psi}{dx^2} + 2a^2\operatorname{sech}^2(ax)\psi(x) = -\frac{2mE}{\hbar^2}\psi(x) \tag{1}$$

Use k for $\sqrt{2mE}/\hbar$ and bring all terms to the left side.

$$\frac{d^2\psi}{dx^2} + [k^2 + 2a^2\operatorname{sech}^2(ax)]\psi(x) = 0$$

Observe from the graph in part (a) that the potential energy has a dip around $x = 0$ and goes to zero as $|x|$ becomes large. Because $V(x)$ is similar to the delta-function well, we expect there to be plane wave solutions, $C_5 e^{-ikx}$ and $C_6 e^{ikx}$, when $|x|$ is large. But unlike the delta-function well, $V(x)$ is a bounded and continuous function, meaning $\psi(x)$ has the same formula for $x < 0$ and $x > 0$. That means the incident and transmitted probability currents are equal, or $T = 1$ and $R = 0$. Check for a solution of the form $\psi(x) = q(x)e^{ikx}$.

$$\frac{d^2}{dx^2}[q(x)e^{ikx}] + [k^2 + 2a^2\operatorname{sech}^2(ax)][q(x)e^{ikx}] = 0$$

Simplify the left side.

$$e^{ikx}\left[\frac{d^2q}{dx^2}+2ik\frac{dq}{dx}+2a^2\operatorname{sech}^2(ax)q(x)\right]=0$$

Divide both sides by e^{ikx}.

$$\frac{d^2q}{dx^2}+2ik\frac{dq}{dx}+2a^2\operatorname{sech}^2(ax)q(x)=0$$

Make the substitution $z=ax$. The derivatives become

$$\frac{dq}{dx}=a\frac{dq}{dz}$$

$$\frac{d^2q}{dx^2}=a^2\frac{d^2q}{dz^2}.$$

As a result,

$$a^2\frac{d^2q}{dz^2}+2iak\frac{dq}{dz}+2a^2\operatorname{sech}^2(z)q(z)=0.$$

Divide both sides by a^2.

$$\frac{d^2q}{dz^2}+\frac{2ik}{a}\frac{dq}{dz}+2\operatorname{sech}^2(z)q(z)=0 \tag{5}$$

This ODE is signficantly harder to solve because of the presence of the first derivative, but that's okay. Since $V(x)=0>-(\hbar^2a^2/m)\operatorname{sech}^2(ax)$ for all finite x, the eigenfunction corresponding to the zero eigenvalue is a scattering state. Whatever formula we find for $q(x)$ should then simplify to this result for the case that $E=0$. The eigenfunction corresponding to the zero eigenvalue therefore serves as a hint for $q(x)$. Plug in a trial solution of the form $q(z)=B_0+B_1\tanh z+B_2z\tanh z$ into equation (5).

$$\frac{d^2}{dz^2}(B_0+B_1\tanh z+B_2z\tanh z)+\frac{2ik}{a}\frac{d}{dz}(B_0+B_1\tanh z+B_2z\tanh z)$$
$$+2\operatorname{sech}^2(z)(B_0+B_1\tanh z+B_2z\tanh z)=0$$

Simplify the left side.

$$\frac{2}{a}(B_0a+B_2a+ikB_1)\operatorname{sech}^2 z+\frac{2ik}{a}B_2z\operatorname{sech}^2 z+\frac{2ik}{a}B_2\tanh z=0$$

Match the coefficients on both sides.

$$B_0a+B_2a+ikB_1=0$$
$$\frac{2ik}{a}B_2=0$$
$$\frac{2ik}{a}B_2=0$$

Set $B_2=0$ to satisfy these last two equations. The first equation then reduces to $B_0a+ikB_1=0$. Choose $B_0=ik$ and $B_1=-a$. As a result, one solution to equation (5) is

$$q(z)=ik-a\tanh z.$$

Use the method of reduction of order now to determine the general solution: Plug in $q(z)=(ik-a\tanh z)p(z)$ into equation (5).

$$\frac{d^2}{dz^2}[(ik-a\tanh z)p(z)]+\frac{2ik}{a}\frac{d}{dz}[(ik-a\tanh z)p(z)]+2\operatorname{sech}^2(z)[(ik-a\tanh z)p(z)]=0$$

Simplify the left side.

$$(ik - a\tanh z)\frac{d^2p}{dz^2} - \frac{2}{a}\left[a^2\operatorname{sech}^2 z + k(k + ia\tanh z)\right]\frac{dp}{dz} = 0$$

This is a first-order linear ODE for dp/dz, so it can be solved with an integrating factor. The general solution is

$$p(z) = C_5 + C_6 e^{-2ikz/a}\frac{ik\cosh z + a\sinh z}{k\cosh z + ia\sinh z}.$$

That means

$$q(z) = (ik - a\tanh z)p(z) = C_5(ik - a\tanh z) + C_6 e^{-2ikz/a}(ia\tanh z - k).$$

Change back to the original variable x.

$$q(x) = C_5[ik - a\tanh(ax)] + C_6 e^{-2ikx}[ia\tanh(ax) - k]$$

Consequently,

$$\begin{aligned}\psi(x) &= q(x)e^{ikx} \\ &= C_5[ik - a\tanh(ax)]e^{ikx} + C_6 e^{-2ikx}[ia\tanh(ax) - k]e^{ikx} \\ &= D_5[ik - a\tanh(ax)]e^{ikx} + D_6[ik + a\tanh(ax)]e^{-ikx}.\end{aligned}$$

Since neither term blows up as $x \to \pm\infty$, both of them must be kept. Solving the ODE in t yields $\phi(t) = e^{-iEt/\hbar}$. Therefore, another solution to Schrödinger's equation is

$$\begin{aligned}\Psi(x,t) &= \phi(x)\phi(t) \\ &= D_5[ik - a\tanh(ax)]e^{ikx}e^{-iEt/\hbar} + D_6[ik + a\tanh(ax)]e^{-ikx}e^{-iEt/\hbar} \\ &= D_5[ik - a\tanh(ax)]\exp\left(ikx - \frac{iEt}{\hbar}\right) + D_6[ik + a\tanh(ax)]\exp\left(-ikx - \frac{iEt}{\hbar}\right)\end{aligned}$$

$$\boxed{\Psi(x,t) = D_5[ik - a\tanh(ax)]\exp\left[ik\left(x - \frac{Et}{\hbar k}\right)\right] + D_6[ik + a\tanh(ax)]\exp\left[-ik\left(x + \frac{Et}{\hbar k}\right)\right],}$$

where $E > 0$ and $k = \sqrt{2mE}/\hbar$. If $|x|$ is large, then this is a linear combination of plane waves travelling to the right and to the left, respectively, as predicted. For a plane wave incident from the left only, D_6 is set to zero; and for a plane wave incident from the right only, D_5 is set to zero.

Again, it must be emphasized that $V(x) = -(\hbar^2 a^2/m)\operatorname{sech}^2(ax)$ is a reflectionless potential because $\psi(x)$ has the same formula for all x. And this is due to the fact that $V(x)$ is bounded and continuous unlike all the cases considered previously in the chapter.