Problem 2.54

The transfer matrix.[64] The S-matrix (Problem 2.53) tells you the *outgoing* amplitudes (B and F) in terms of the *incoming* amplitudes (A and G)—Equation 2.180. For some purposes it is more convenient to work with the **transfer matrix**, M, which gives you the amplitudes to the *right* of the potential (F and G) in terms of those to the *left* (A and B):

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{2.183}$$

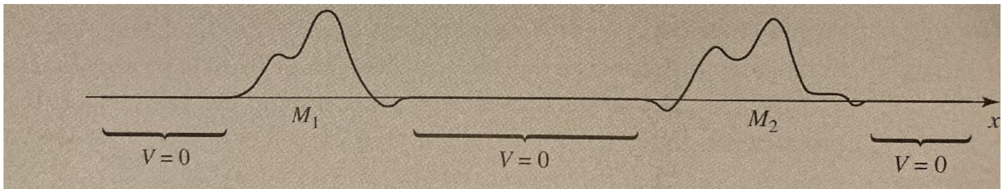


Figure 2.22: A potential consisting of two isolated pieces (Problem 2.54).

(a) Find the four elements of the M-matrix, in terms of the elements of the S-matrix, and vice-versa. Express R_l, T_l, R_r, and T_r (Equations 2.181 and 2.182) in terms of elements of the M-matrix.

(b) Suppose you have a potential consisting of two isolated pieces (Figure 2.22). Show that the M-matrix for the combination is the *product* of the two M-matrices for each section separately:

$$\mathsf{M} = \mathsf{M}_2 \mathsf{M}_1. \tag{2.184}$$

(This obviously generalizes to any number of pieces, and accounts for the usefulness of the M-matrix.)

(c) Construct the M-matrix for scattering from a single delta-function potential at point a:

$$V(x) = -\alpha\delta(x - a).$$

(d) By the method of part (b), find the M-matrix for scattering from the double-delta function

$$V(x) = -\alpha[\delta(x + a) + \delta(x - a)].$$

What is the transmission coefficient for this potential?

Solution

[64]For applications of this method see, for instance, D. J. Griffiths and C. A. Steinke, *Am. J. Phys.* **69**, 137 (2001) or S. Das, *Am. J. Phys.* **83**, 590 (2015).

Part (a)

The scattering and transfer matrices are defined by

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Write out the equations implied from these matrix equations.

$$\begin{aligned} B &= S_{11}A + S_{12}G \qquad & F &= M_{11}A + M_{12}B \\ F &= S_{21}A + S_{22}G \qquad & G &= M_{21}A + M_{22}B \end{aligned}$$

Solve these two equations on the left for F and G, and solve these two equations on the right for B and F.

$$\begin{aligned} F &= -\frac{S_{11}S_{22} - S_{12}S_{21}}{S_{12}}A + \frac{S_{22}}{S_{12}}B \qquad & B &= -\frac{M_{21}}{M_{22}}A + \frac{1}{M_{22}}G \\ G &= -\frac{S_{11}}{S_{12}}A + \frac{1}{S_{12}}B \qquad & F &= \frac{M_{11}M_{22} - M_{12}M_{21}}{M_{22}}A + \frac{M_{12}}{M_{22}}G \end{aligned}$$

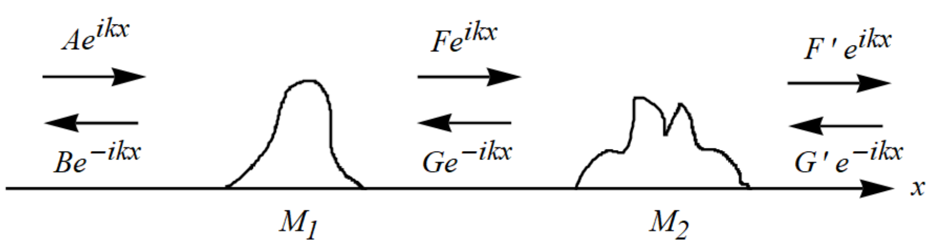
Therefore, the transfer and scattering matrices in terms of the scattering elements and transfer elements, respectively, are

$$\mathsf{M} = \begin{pmatrix} -\dfrac{S_{11}S_{22} - S_{12}S_{21}}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ -\dfrac{S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{pmatrix} \qquad \mathsf{S} = \begin{pmatrix} -\dfrac{M_{21}}{M_{22}} & \dfrac{1}{M_{22}} \\ \dfrac{M_{11}M_{22} - M_{12}M_{21}}{M_{22}} & \dfrac{M_{12}}{M_{22}} \end{pmatrix}.$$

Based on equation (2.181) and equation (2.182), the formulas for the reflection and transmission coefficients are

$$\begin{aligned} R_l &= \left|-\frac{M_{21}}{M_{22}}\right|^2 = \left|\frac{M_{21}}{M_{22}}\right|^2 \qquad & T_l &= \left|\frac{M_{11}M_{22} - M_{12}M_{21}}{M_{22}}\right|^2 \\ R_r &= \left|\frac{M_{12}}{M_{22}}\right|^2 \qquad & T_r &= \left|\frac{1}{M_{22}}\right|^2. \end{aligned}$$

Part (b)



The outgoing and incoming amplitudes can be related by means of the transfer matrices.

$$\begin{pmatrix} F \\ G \end{pmatrix} = \mathsf{M}_1 \begin{pmatrix} A \\ B \end{pmatrix} \qquad \begin{pmatrix} F' \\ G' \end{pmatrix} = \mathsf{M}_2 \begin{pmatrix} F \\ G \end{pmatrix}$$

Substitute this first matrix equation into the second one.

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \mathsf{M}_2 \left[\mathsf{M}_1 \begin{pmatrix} A \\ B \end{pmatrix} \right] = \mathsf{M}_2 \mathsf{M}_1 \begin{pmatrix} A \\ B \end{pmatrix}$$

Therefore, the two potentials can be lumped together by using the effective transfer matrix,

$$\mathsf{M} = \mathsf{M}_2 \mathsf{M}_1.$$

Part (c)

With

$$V(x) = -\alpha\delta(x-a),$$

the Schrödinger equation becomes

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \alpha\delta(x-a)\Psi(x,t), \quad -\infty < x < \infty, \ t > 0.$$

Applying the method of separation of variables [$\Psi(x,t) = \psi(x)\phi(t)$] results in two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar \frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \alpha\delta(x-a) &= E \end{aligned}\right\}$$

Solve the TISE for the second derivative.

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}[\alpha\delta(x-a) + E]\psi(x) \tag{1}$$

The delta function is zero everywhere except $x = a$.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x), \quad x \neq a$$

For scattering states, $E > 0$, which means the general solution is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < a \\ Fe^{ikx} + Ge^{-ikx} & \text{if } x > a \end{cases},$$

where $k = \sqrt{2mE}/\hbar$. The wave function [and consequently $\psi(x)$] is required to be continuous at $x = a$.

$$\lim_{x \to a^-} \psi(x) = \lim_{x \to a^+} \psi(x) : \quad Ae^{ika} + Be^{-ika} = Fe^{ika} + Ge^{-ika} \tag{2}$$

Integrate both sides of equation (1) with respect to x from $a - \epsilon$ to $a + \epsilon$, where ϵ is a really small positive number.

$$\begin{aligned} \int_{a-\epsilon}^{a+\epsilon} \frac{d^2\psi}{dx^2}\,dx &= -\frac{2m}{\hbar^2}\int_{a-\epsilon}^{a+\epsilon} [\alpha\delta(x-a) + E]\psi(x)\,dx \\ \left.\frac{d\psi}{dx}\right|_{a-\epsilon}^{a+\epsilon} &= -\frac{2m}{\hbar^2}\left[\alpha\int_{a-\epsilon}^{a+\epsilon} \delta(x-a)\psi(x)\,dx + E\int_{a-\epsilon}^{a+\epsilon} \psi(x)\,dx\right] \\ &= -\frac{2m}{\hbar^2}\left[\alpha\psi(a) + E\psi(a)\int_{a-\epsilon}^{a+\epsilon} dx\right] \\ &= -\frac{2m}{\hbar^2}\left[\alpha\psi(a) + E\psi(a)(2\epsilon)\right] \end{aligned}$$

Take the limit as $\epsilon \to 0$.

$$\left.\frac{d\psi}{dx}\right|_{a^-}^{a^+} = -\frac{2m\alpha}{\hbar^2}\psi(a)$$

The spatial derivative of the wave function, on the other hand, is discontinuous at $x = a$.

$$\lim_{x\to a^+}\frac{d\psi}{dx} - \lim_{x\to a^-}\frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2}\psi(a):$$

$$ik(Fe^{ika} - Ge^{-ika}) - ik(Ae^{ika} - Be^{-ika}) = -\frac{2m\alpha}{\hbar^2}(Ae^{ika} + Be^{-ika}) \quad (3)$$

Solve equations (2) and (3) for F and G.

$$F = \frac{k\hbar^2 + im\alpha}{k\hbar^2}A + \frac{im\alpha e^{-2ika}}{k\hbar^2}B$$

$$G = -\frac{im\alpha e^{2ika}}{k\hbar^2}A + \frac{k\hbar^2 - im\alpha}{k\hbar^2}B$$

Therefore, the transfer matrix for the delta-function well centered at $x = a$ is

$$\mathsf{M} = \begin{pmatrix} \frac{k\hbar^2 + im\alpha}{k\hbar^2} & \frac{im\alpha e^{-2ika}}{k\hbar^2} \\ -\frac{im\alpha e^{2ika}}{k\hbar^2} & \frac{k\hbar^2 - im\alpha}{k\hbar^2} \end{pmatrix}.$$

Part (d)

The transfer matrix for the delta-function well centered at $x = -a$ is

$$\mathsf{M}_1 = \begin{pmatrix} \frac{k\hbar^2 + im\alpha}{k\hbar^2} & \frac{im\alpha e^{2ika}}{k\hbar^2} \\ -\frac{im\alpha e^{-2ika}}{k\hbar^2} & \frac{k\hbar^2 - im\alpha}{k\hbar^2} \end{pmatrix},$$

and the one found for the well centered at $x = a$ is M_2. The transfer matrix for $V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$ is then

$$\mathsf{M} = \mathsf{M}_2\mathsf{M}_1 = \begin{pmatrix} \frac{k\hbar^2 + im\alpha}{k\hbar^2} & \frac{im\alpha e^{-2ika}}{k\hbar^2} \\ -\frac{im\alpha e^{2ika}}{k\hbar^2} & \frac{k\hbar^2 - im\alpha}{k\hbar^2} \end{pmatrix} \times \begin{pmatrix} \frac{k\hbar^2 + im\alpha}{k\hbar^2} & \frac{im\alpha e^{2ika}}{k\hbar^2} \\ -\frac{im\alpha e^{-2ika}}{k\hbar^2} & \frac{k\hbar^2 - im\alpha}{k\hbar^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(k\hbar^2 + im\alpha)^2 + m^2\alpha^2 e^{-4ika}}{k^2\hbar^4} & \frac{im\alpha e^{2ika}(k\hbar^2 + im\alpha) + im\alpha e^{-2ika}(k\hbar^2 - im\alpha)}{k^2\hbar^4} \\ \frac{-im\alpha e^{2ika}(k\hbar^2 + im\alpha) - im\alpha e^{-2ika}(k\hbar^2 - im\alpha)}{k^2\hbar^4} & \frac{m^2\alpha^2 e^{4ika} + (k\hbar^2 - im\alpha)^2}{k^2\hbar^4} \end{pmatrix}.$$

Therefore,

$$\mathsf{M} = \begin{pmatrix} \dfrac{(k\hbar^2 + im\alpha)^2 + m^2\alpha^2 e^{-4ika}}{k^2\hbar^4} & \dfrac{2im\alpha(k\hbar^2\cos 2ka - m\alpha\sin 2ka)}{k^2\hbar^4} \\ -\dfrac{2im\alpha(k\hbar^2\cos 2ka - m\alpha\sin 2ka)}{k^2\hbar^4} & \dfrac{(k\hbar^2 - im\alpha)^2 + m^2\alpha^2 e^{4ika}}{k^2\hbar^4} \end{pmatrix}.$$

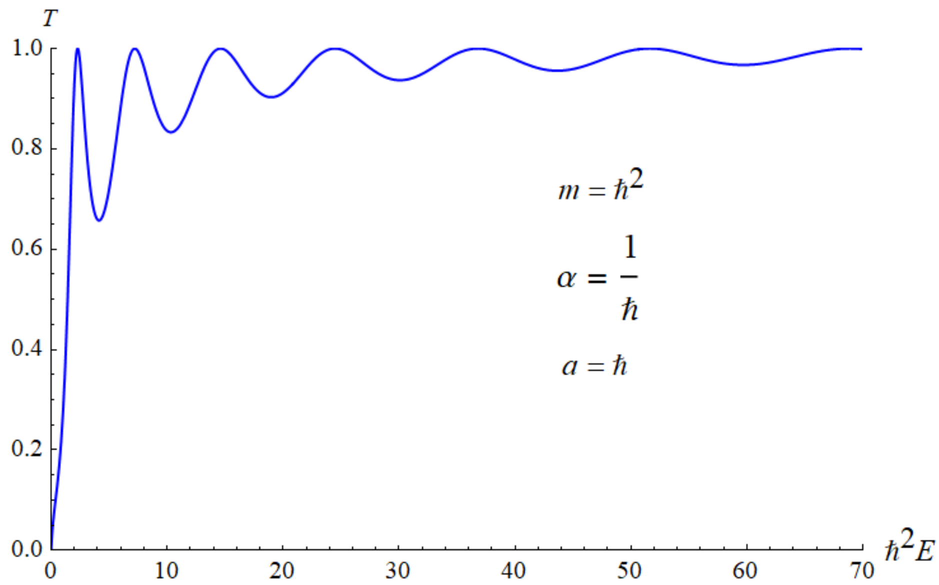
Note that the determinant of this matrix is 1: $\det \mathsf{M} = \mathsf{M}_{11}\mathsf{M}_{22} - \mathsf{M}_{12}\mathsf{M}_{21} = 1$. Consequently, whether the scattering is from the left or right, the formulas at the end of part (a) give the same answer.

$$T_l = \left|\frac{M_{11}M_{22} - M_{12}M_{21}}{M_{22}}\right|^2 = \left|\frac{1}{M_{22}}\right|^2 = T_r$$

The transmission coefficient for this potential is therefore

$$\begin{aligned}
T &= \left|\frac{1}{M_{22}}\right|^2 \\
&= \left|\frac{k^2\hbar^4}{(k\hbar^2 - im\alpha)^2 + m^2\alpha^2 e^{4ika}}\right|^2 \\
&= \frac{k^2\hbar^4}{(k\hbar^2 + im\alpha)^2 + m^2\alpha^2 e^{-4ika}} \cdot \frac{k^2\hbar^4}{(k\hbar^2 - im\alpha)^2 + m^2\alpha^2 e^{4ika}} \\
&= \frac{k^4\hbar^8}{(k\hbar^2 + im\alpha)^2(k\hbar^2 - im\alpha)^2 + m^2\alpha^2 e^{4ika}(k\hbar^2 + im\alpha)^2 + m^2\alpha^2 e^{-4ika}(k\hbar^2 - im\alpha)^2 + m^4\alpha^4} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 2k^2m^2\alpha^2\hbar^4 + 2m^4\alpha^4 - m^4\alpha^4(e^{4ika} + e^{-4ika}) + 2ikm^3\alpha^3\hbar^2(e^{4ika} - e^{-4ika}) + k^2m^2\alpha^2\hbar^4(e^{4ika} + e^{-4ika})} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 2k^2m^2\alpha^2\hbar^4 + 2m^4\alpha^4 - m^4\alpha^4(2\cos 4ka) + 2ikm^3\alpha^3\hbar^2(2i\sin 4ka) + k^2m^2\alpha^2\hbar^4(2\cos 4ka)} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 2k^2m^2\alpha^2\hbar^4(1 + \cos 4ka) + 2m^4\alpha^4(1 - \cos 4ka) - 4km^3\alpha^3\hbar^2\sin 4ka} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 2k^2m^2\alpha^2\hbar^4(2\cos^2 2ka) + 2m^4\alpha^4(2\sin^2 2ka) - 4km^3\alpha^3\hbar^2(2\sin 2ka\cos 2ka)} \\
&= \frac{k^4\hbar^8}{k^4\hbar^8 + 4m^2\alpha^2\left(k^2\hbar^4\cos^2 2ka - 2km\alpha\hbar^2\sin 2ka\cos 2ka + m^2\alpha^2\sin^2 2ka\right)} \\
&= \frac{1}{1 + \frac{4m^2\alpha^2}{k^2\hbar^4}\left(\cos^2 2ka - \frac{2m\alpha}{k\hbar^2}\sin 2ka\cos 2ka + \frac{m^2\alpha^2}{k^2\hbar^4}\sin^2 2ka\right)} \\
&= \frac{1}{1 + \frac{4m^2\alpha^2}{k^2\hbar^4}\left(\cos 2ka - \frac{m\alpha}{k\hbar^2}\sin 2ka\right)^2} \\
&= \frac{1}{1 + \frac{2m\alpha^2}{\hbar^2 E}\left(\cos\frac{2a\sqrt{2mE}}{\hbar} - \frac{m\alpha}{\hbar\sqrt{2mE}}\sin\frac{2a\sqrt{2mE}}{\hbar}\right)^2}.
\end{aligned}$$

Below is a plot of the transmission coefficient T versus $\hbar^2 E$ for the special case that $m = \hbar^2$, $a = \hbar$, and $\alpha = 1/\hbar$.



This result is in agreement with the one in Problem 2.28.