Problem 2.55

Find the ground state energy of the harmonic oscillator, to five significant digits, by the "wag-the-dog" method. That is, solve Equation 2.73, varying K until you get a wave function that goes to zero at large ξ. In Mathematica, appropriate input code would be

```
Plot[
  Evaluate[
    u[x]/.
      NDSolve[
        {u''[x] - (x^2 - K)*u[x] == 0, u[0] == 1, u'[0] == 0}, u[x], {x, 0, b}
      ]
    ],
  {x, a, b}, PlotRange -> {c, d}
]
```

(Here (a, b) is the horizontal range of the graph, and (c, d) is the vertical range—start with $a = 0$, $b = 10$, $c = -10$, $d = 10$.) We know that the correct solution is $K = 2n + 1$, so you might start with a "guess" of $K = 0.9$. Notice what the "tail" of the wave function does. Now try $K = 1.1$, and note that the tail flips over. Somewhere in between those values lies the correct solution. Zero in on it by bracketing K tighter and tighter. As you do so, you may want to adjust a, b, c, and d, to zero in on the cross-over point.

Solution

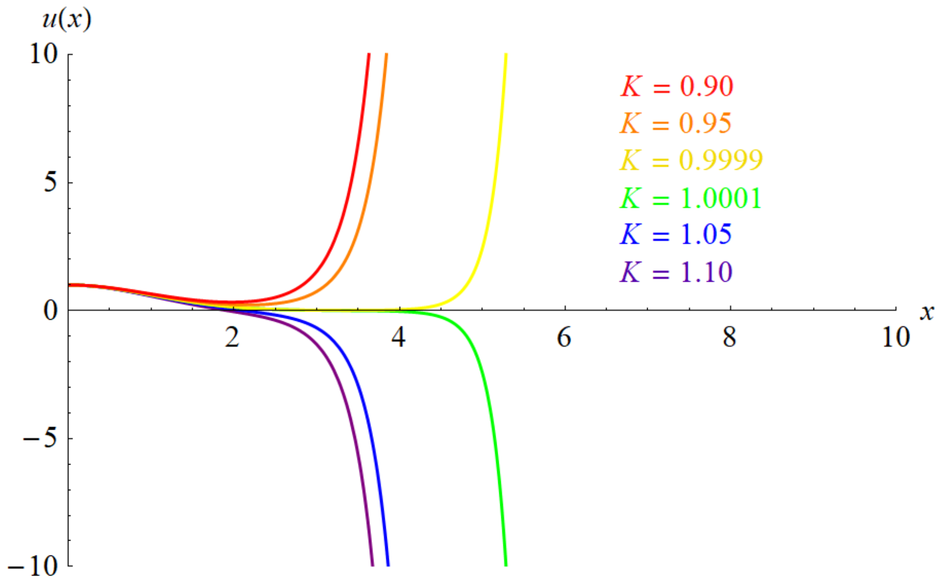
Rather than plugging in one value for K at a time, it's more convenient to use the **Manipulate[]** function in Mathematica. It places a horizontal scroll bar above the graph that automatically adjusts the value of K and plots the result. By the way, the character 'K' is reserved in Mathematica, so a different letter should be chosen. The preferable code to use is as follows.

```
Manipulate[
  Plot[
    Evaluate[
      u[x]/.
        NDSolve[
          {u''[x] - (x^2 - L)*u[x] == 0, u[0] == 1, u'[0] == 0}, u[x], {x, 0, 10}
        ]
      ],
    {x, 0, 10}, PlotRange -> {-10, 10}
  ], {L, 0.9, 1.1}
]
```

Below are plots of the solution to the boundary value problem,

$$u'' - (x^2 - K)u = 0, \qquad u(0) = 1,\ u'(0) = 0,$$

for several values of K.



Because the graph flips over when using $K = 0.9$ and $K = 1.1$, there's an eigenvalue somewhere within $0.9 < K < 1.1$. The graph also flips over when using $K = 0.95$ and $K = 1.05$, so the range narrows to $0.95 < K < 1.05$. It does the same using $K = 0.9999$ and $K = 1.0001$, so the range narrows further to $0.9999 < K < 1.0001$. To five significant figures, then,

$$K \approx 1.0000,$$

which means the ground state energy of the harmonic oscillator is

$$K = \frac{2E}{\hbar\omega} \approx 1.0000 \quad \rightarrow \quad E \approx 0.50000\hbar\omega.$$