Problem 2.59

The "bouncing ball."[65] Suppose

$$V(x) = \begin{cases} mgx, & x > 0, \\ \infty, & x \leq 0. \end{cases} \tag{2.185}$$

(a) Solve the (time-independent) Schrödinger equation for this potential. *Hint:* First convert it to dimensionless form:

$$-y''(z) + zy(z) = \epsilon y(z) \tag{2.186}$$

by letting $z \equiv ax$ and $y(z) = (1/\sqrt{a})\,\psi(x)$ (the $\sqrt{a}$ is just so $y(z)$ is normalized with respect to z when $\psi(x)$ is normalized with respect to x.) What are the constants a and ϵ? Actually, we might as well set $a \to 1$—this amounts to a convenient choice for the unit of length. Find the general solution to this equation (in Mathematica **DSolve** will do the job). The result is (of course) a linear combination of two (probably unfamiliar) functions. Plot each of them, for $(-15 < z < 5)$. One of them clearly does not go to zero at large z (more precisely, it's not normalizable), so discard it. The allowed values of ϵ (and hence of E) are determined by the condition $\psi(0) = 0$. Find the ground state ϵ_1 numerically (in Mathematica **FindRoot** will do it), and also the 10th, ϵ_{10}. Obtain the corresponding normalization factors. Plot $\psi_1(x)$ and $\psi_{10}(x)$, for $0 \leq z < 16$. Just as a check, confirm that $\psi_1(x)$ and $\psi_{10}(x)$ are orthogonal.

(b) Find (numerically) the uncertainties σ_x and σ_p for these two states, and check that the uncertainty principle is obeyed.

(c) The probability of finding the ball in the neighborhood dx of height x is (of course) $\rho_Q(x)\,dx = |\psi(x)|^2\,dx$. The nearest classical analog would be the fraction of *time* an elastically bouncing ball (with the same energy, E) spends in the neighborhood dx of height x (see Problem 1.11). Show that this is

$$\rho_C(x)\,dx = \frac{mg}{2\sqrt{E(E - mgx)}}\,dx. \tag{2.187}$$

or, in our units (with $a = 1$),

$$\rho_C(x) = \frac{1}{2\sqrt{\epsilon(\epsilon - x)}}. \tag{2.188}$$

Plot $\rho_Q(x)$ and $\rho_C(x)$ for the state $\psi_{10}(x)$, on the range $0 \leq x \leq 12.5$; superimpose the graphs (**Show**, in Mathematica), and comment on the result.

Solution

[65] This problem was suggested by Nicholas Wheeler.

Part (a)

The Schrödinger equation governs the time evolution of the wave function.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

As the particle cannot go through the ground at $x = 0$, the potential energy function is

$$V(x,t) = \begin{cases} mgx & \text{if } x > 0 \\ \infty & \text{if } x \leq 0 \end{cases}.$$

Split up the PDE over the intervals where $V(x,t)$ is defined.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + (\infty)\Psi(x,t), \quad x \leq 0; \qquad i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mgx\Psi(x,t), \quad x > 0$$

Only the trivial solution, $\Psi(x,t) = 0$, satisfies Schrödinger's equation on $x \leq 0$. Because the wave function must be continuous, $\Psi(0,t) = 0$ is a boundary condition for the remaining PDE on $x > 0$.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mgx\Psi(x,t), \quad x > 0,\ t > 0$$

$$\Psi(0,t) = 0$$

Since this PDE and its associated boundary conditions (also $\Psi \to 0$ as $x \to \infty$) are linear and homogeneous, the method of separation of variables can be applied to solve it successfully. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] + mgx[\psi(x)\phi(t)]$$

and the boundary condition at $x = 0$.

$$\Psi(0,t) = 0 \quad \to \quad \psi(0)\phi(t) = 0 \quad \to \quad \psi(0) = 0$$

Evaluate the derivatives in the PDE.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) + mgx\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ in order to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + mgx$$

The only way a function of t can be equal to a function of x is if both are equal to a constant.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + mgx = E$$

As a result of separating variables, Schrödinger's equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + mgx &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions are satisfied are called the eigenvalues (or eigenenergies in this context), and the nontrivial solutions associated with them are called the eigenfunctions (or eigenstates in this context). The ODE in x is known as the time-independent Schrödinger equation (TISE) and can be written as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi(x) = E\psi(x).$$

The aim here is to make it dimensionless. Notice that each of the terms has dimensions of energy. Divide both sides by mg.

$$-\frac{\hbar^2}{2m^2g}\frac{d^2\psi}{dx^2} + x\psi(x) = \frac{E}{mg}\psi(x)$$

Now the terms have dimensions of length. Divide both sides by x_0, a constant with dimensions of length, so that each of the terms is dimensionless.

$$-\frac{\hbar^2}{2m^2gx_0}\frac{d^2\psi}{dx^2} + \frac{x}{x_0}\psi(x) = \frac{E}{mgx_0}\psi(x) \tag{1}$$

Introduce the dimensionless position variable z, defined by

$$z = \frac{x}{x_0}.$$

Use the chain rule to write the second derivative of ψ in terms of this new variable.

$$\frac{d\psi}{dx} = \frac{d\psi}{dz}\frac{dz}{dx} = \frac{d\psi}{dz}\frac{1}{x_0}$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left(\frac{d\psi}{dz}\frac{1}{x_0}\right) = \frac{1}{x_0^2}\frac{d^2\psi}{dz^2}$$

Consequently, equation (1) becomes

$$-\frac{\hbar^2}{2m^2gx_0^3}\frac{d^2\psi}{dz^2} + z\psi(z) = \frac{E}{mgx_0}\psi(z). \tag{2}$$

x_0 can be conveniently chosen so that the coefficient of the second derivative disappears.

$$\frac{\hbar^2}{2m^2gx_0^3} = 1$$

Solve for x_0.

$$x_0 = \sqrt[3]{\frac{\hbar^2}{2m^2g}}$$

Introduce the dimensionless energy variable ϵ, defined by

$$\epsilon = \frac{E}{mgx_0} = \frac{E}{mg\sqrt[3]{\frac{\hbar^2}{2m^2g}}} = \frac{E}{\sqrt[3]{\frac{\hbar^2mg^2}{2}}} = \sqrt[3]{\frac{2}{\hbar^2mg^2}}E.$$

Consequently, equation (2) becomes

$$-\frac{d^2\psi}{dz^2} + z\psi(z) = \epsilon\psi(z). \tag{3}$$

By making the change of variables, $z = x/x_0$, the normalization condition becomes

$$1 = \int_0^\infty [\psi(x)]^2\,dx = \int_0^\infty \{\psi[x(z)]\}^2(x_0\,dz) = \int_0^\infty \left[\sqrt{x_0}\psi(z)\right]^2 dz.$$

To turn it back to its usual form, make the change of variables,

$$y(z) = \sqrt{x_0}\psi(z).$$

Then

$$1 = \int_0^\infty [y(z)]^2\,dz,$$

and equation (3) becomes

$$-\frac{d^2}{dz^2}\left[\frac{y(z)}{\sqrt{x_0}}\right] + z\left[\frac{y(z)}{\sqrt{x_0}}\right] = \epsilon\left[\frac{y(z)}{\sqrt{x_0}}\right]$$

$$-\frac{1}{\sqrt{x_0}}\frac{d^2y}{dz^2} + z\left[\frac{y(z)}{\sqrt{x_0}}\right] = \epsilon\left[\frac{y(z)}{\sqrt{x_0}}\right].$$

Finally, multiply both sides by $\sqrt{x_0}$ to get the dimensionless TISE.

$$-\frac{d^2y}{dz^2} + zy(z) = \epsilon y(z)$$

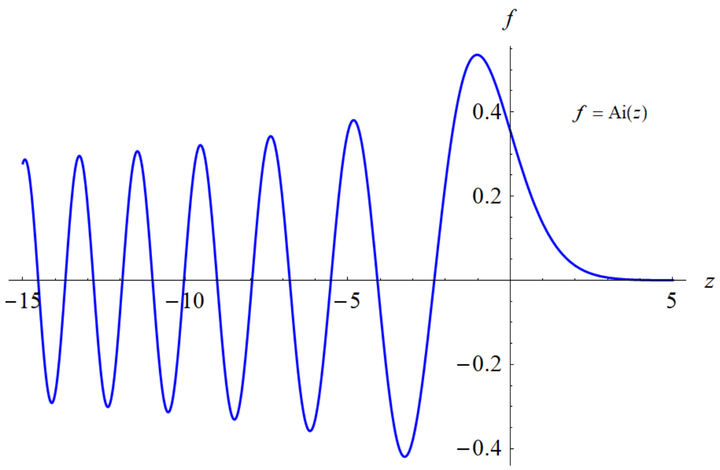
Rewrite it as

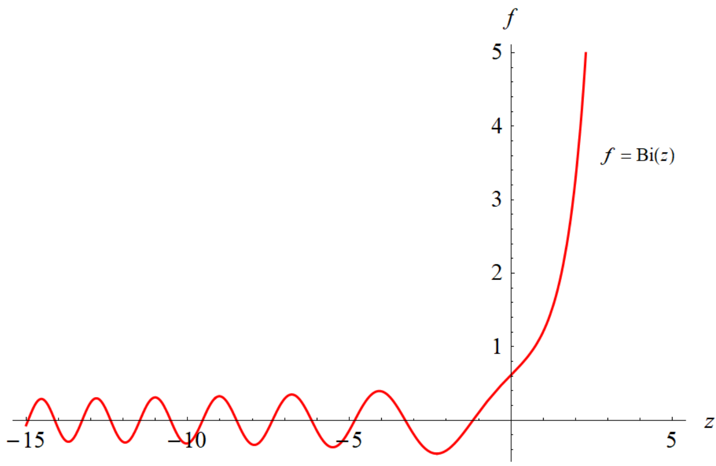
$$\frac{d^2y}{dz^2} = (z-\epsilon)y.$$

The general solution to this ODE can be written in terms of the Airy functions, $\text{Ai}(z)$ and $\text{Bi}(z)$. (Note that the Airy equation is $y'' = zy$.)

$$y(z) = C_1\,\text{Ai}(z-\epsilon) + C_2\,\text{Bi}(z-\epsilon)$$

Below are plots of the Airy functions over $-15 < z < 5$.





The boundary conditions associated with the dimensionless TISE are $y(0) = 0$ and $y(\infty) = 0$. The graph of $\text{Bi}(z)$ indicates that it diverges as $z \to \infty$, so set $C_2 = 0$ to satisfy the latter.

$$y(z) = C_1 \,\text{Ai}(z - \epsilon)$$

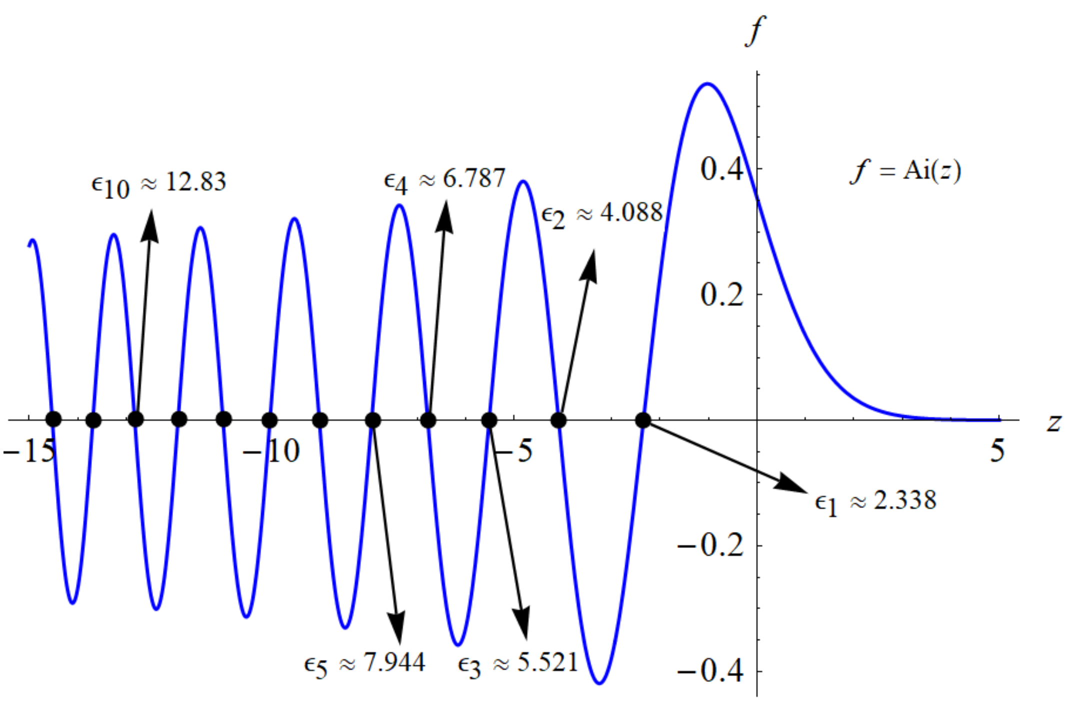
Apply the former boundary condition now.

$$y(0) = C_1 \,\text{Ai}(-\epsilon) = 0$$

To avoid the trivial solution, insist that $C_1 \neq 0$.

$$\text{Ai}(-\epsilon) = 0$$

The dimensionless energies are therefore the zeros (without the minus signs) of $\text{Ai}(z)$.



The nth eigenfunction is then

$$y_n(z) = C_1 \,\text{Ai}(z - \epsilon_n), \quad n \geq 1.$$

C_1 is arbitrary and is chosen so that the integral of $[y_n(z)]^2$ over $0 < z < \infty$ is 1. It's different for each eigenfunction.

$$1 = \int_0^\infty [y_n(z)]^2 \, dz \quad \Rightarrow \quad \begin{cases} 1 = C_1^2 \displaystyle\int_0^\infty [\text{Ai}(z - \epsilon_1)]^2 \, dz = C_1^2 \int_{-\epsilon_1}^\infty [\text{Ai}(u)]^2 \, du \approx C_1^2(0.4917) \\ 1 = C_1^2 \displaystyle\int_0^\infty [\text{Ai}(z - \epsilon_{10})]^2 \, dz = C_1^2 \int_{-\epsilon_{10}}^\infty [\text{Ai}(u)]^2 \, du \approx C_1^2(1.140) \end{cases}$$

As a result, the normalization constants for $y_1(z)$ and $y_{10}(z)$ are respectively

$$C_1 \approx 1.426 \quad \text{and} \quad C_1 \approx 0.9365.$$

Now that the eigenfunctions are known, change back to the original variables.

$$\sqrt{x_0}\psi_n(x) = C_1 \operatorname{Ai}\left(\frac{x}{x_0} - \epsilon_n\right), \quad n \geq 1$$

Therefore, the first and tenth eigenstates are

$$\psi_1(x) \approx \frac{1.426}{\sqrt{x_0}} \operatorname{Ai}\left(\frac{x}{x_0} - \epsilon_1\right) = 1.426\sqrt[6]{\frac{2m^2g}{\hbar^2}} \operatorname{Ai}\left(\sqrt[3]{\frac{2m^2g}{\hbar^2}}x - \epsilon_1\right)$$

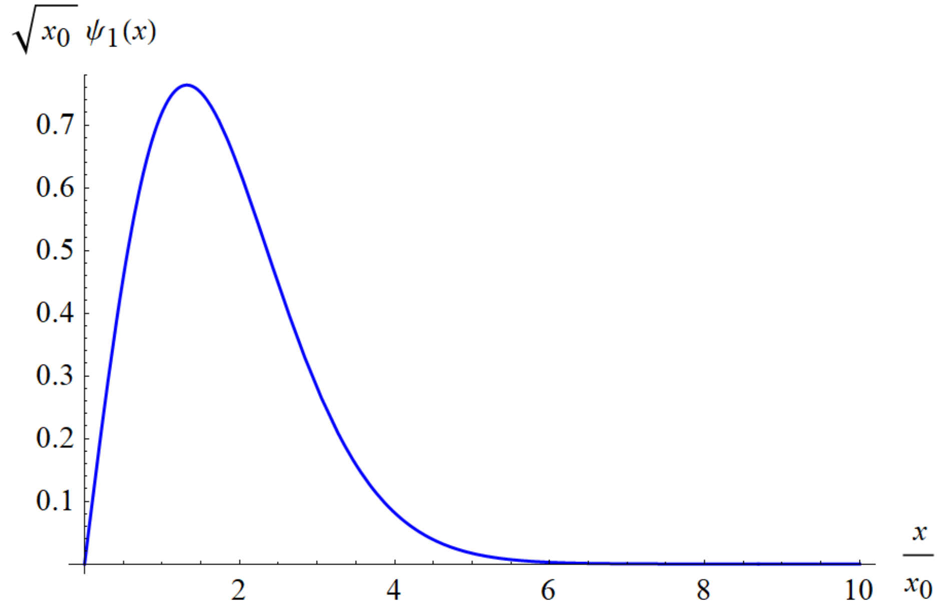
$$\psi_{10}(x) \approx \frac{0.9365}{\sqrt{x_0}} \operatorname{Ai}\left(\frac{x}{x_0} - \epsilon_{10}\right) = 0.9365\sqrt[6]{\frac{2m^2g}{\hbar^2}} \operatorname{Ai}\left(\sqrt[3]{\frac{2m^2g}{\hbar^2}}x - \epsilon_{10}\right),$$

and the corresponding energies are

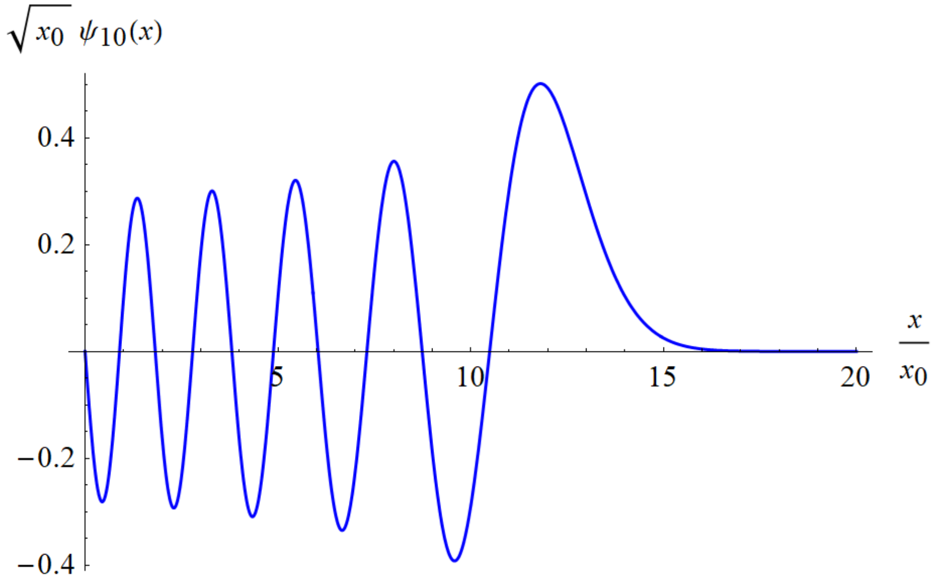
$$\epsilon_1 = \sqrt[3]{\frac{2}{\hbar^2 mg^2}}E_1 \approx 2.338 \qquad \rightarrow \qquad E_1 \approx 1.856\sqrt[3]{\hbar^2 mg^2}$$

$$\epsilon_{10} = \sqrt[3]{\frac{2}{\hbar^2 mg^2}}E_{10} \approx 12.83 \qquad \rightarrow \qquad E_{10} \approx 10.18\sqrt[3]{\hbar^2 mg^2}.$$

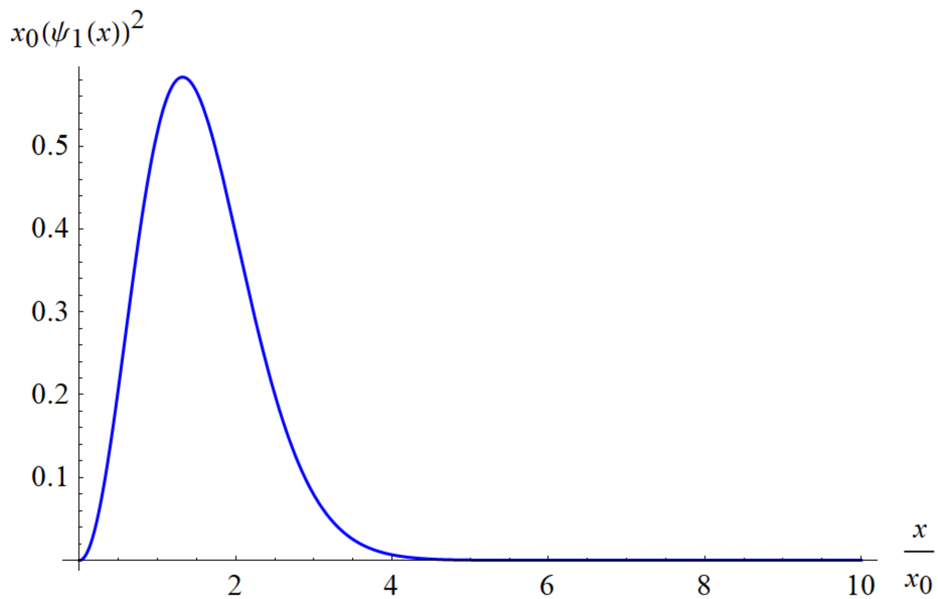
Below is a plot of $\sqrt{x_0}\psi_1(x)$ versus x/x_0.



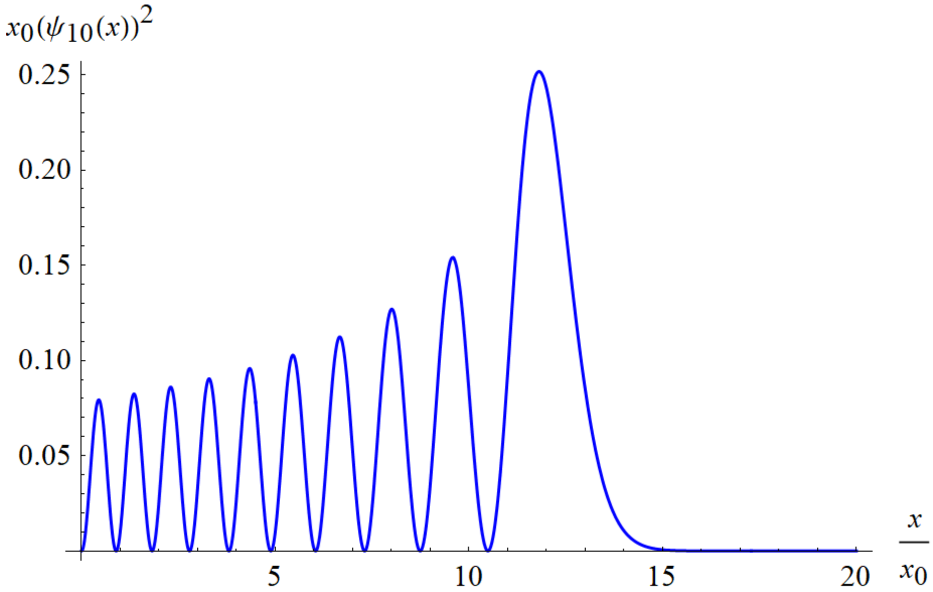
Below is a plot of $\sqrt{x_0}\psi_{10}(x)$ versus x/x_0.



The probability distribution for a particle's position in each of these eigenstates is given by $[\psi_1(x)]^2$ and $[\psi_{10}(x)]^2$, respectively. A plot of $x_0[\psi_1(x)]^2$ is shown below versus x/x_0.



A plot of $x_0[\psi_{10}(x)]^2$ is shown below versus x/x_0.



The inner product of $\psi_1(x)$ and $\psi_{10}(x)$ evaluates to

$$\begin{aligned}\langle \psi_1(x), \psi_{10}(x) \rangle &= \int_0^\infty \psi_1^*(x)\psi_{10}(x)\,dx \\ &\approx \int_0^\infty \left[\frac{1.426}{\sqrt{x_0}}\,\text{Ai}\left(\frac{x}{x_0} - \epsilon_1\right)\right]\left[\frac{0.9365}{\sqrt{x_0}}\,\text{Ai}\left(\frac{x}{x_0} - \epsilon_{10}\right)\right] dx \\ &\approx \frac{1.335}{x_0}\int_0^\infty \text{Ai}\left(\frac{x}{x_0} - 2.338\right)\text{Ai}\left(\frac{x}{x_0} - 12.83\right) dx \\ &\approx \frac{1.335}{x_0}\int_0^\infty \text{Ai}(z - 2.338)\,\text{Ai}(z - 12.83)(x_0\,dz) \\ &\approx 1.335\int_0^\infty \text{Ai}(z - 2.338)\,\text{Ai}(z - 12.83)\,dz \\ &\approx -1.268 \times 10^{-4},\end{aligned}$$

which is quite close to zero. Using more digits for each of the numbers in the integral results in a number that's even smaller, so it's reasonable to say that $\psi_1(x)$ and $\psi_{10}(x)$ are orthogonal.

Part (b)

The solution to the ODE in t is $e^{-iEt/\hbar}$, so the first and tenth stationary states are

$$\Psi_1(x,t) = \psi_1(x)e^{-iE_1t/\hbar}$$
$$\Psi_{10}(x,t) = \psi_{10}(x)e^{-iE_{10}t/\hbar}.$$

Begin by calculating the expectation values for the first stationary state. The expectation value of x at time t is

$$\begin{aligned}\langle x\rangle = \int_0^\infty \Psi_1^*(x,t)(x)\Psi_1(x,t)\,dx = \int_0^\infty x[\psi_1(x)]^2\,dx &\approx \int_0^\infty x\left[\frac{1.426}{\sqrt{x_0}}\,\text{Ai}\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\ &\approx \frac{2.033}{x_0}\int_0^\infty x\left[\text{Ai}\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\ &\approx \frac{2.033}{x_0}\int_0^\infty x_0 z\,[\text{Ai}(z-\epsilon_1)]^2\,(x_0\,dz) \\ &\approx 2.033x_0\int_0^\infty z\,[\text{Ai}(z-\epsilon_1)]^2\,dz \\ &\approx 2.033x_0\int_{-\epsilon_1}^\infty (u+\epsilon_1)\,[\text{Ai}(u)]^2\,du \\ &\approx 1.558x_0.\end{aligned}$$

The expectation value of x^2 at time t is

$$\begin{aligned}\langle x^2\rangle = \int_0^\infty \Psi_1^*(x,t)(x^2)\Psi_1(x,t)\,dx = \int_0^\infty x^2[\psi_1(x)]^2\,dx &\approx \int_0^\infty x^2\left[\frac{1.426}{\sqrt{x_0}}\,\text{Ai}\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\ &\approx \frac{2.033}{x_0}\int_0^\infty x^2\left[\text{Ai}\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\ &\approx \frac{2.033}{x_0}\int_0^\infty x_0^2 z^2\,[\text{Ai}(z-\epsilon_1)]^2\,(x_0\,dz) \\ &\approx 2.033x_0^2\int_0^\infty z^2\,[\text{Ai}(z-\epsilon_1)]^2\,dz \\ &\approx 2.033x_0^2\int_{-\epsilon_1}^\infty (u+\epsilon_1)^2\,[\text{Ai}(u)]^2\,du \\ &\approx 2.915x_0^2.\end{aligned}$$

The standard deviation in x for the first stationary state is then

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \approx \sqrt{2.915x_0^2 - 1.558^2x_0^2} \approx 0.6973x_0.$$

The expectation value of p at time t is

$$\begin{aligned}
\langle p\rangle = \int_0^\infty \Psi_1^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi_1(x,t)\,dx &= -i\hbar\int_0^\infty \psi_1(x)\frac{d\psi_1}{dx}\,dx \\
&= -i\hbar\left\{\underbrace{[\psi_1(x)]^2\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{d\psi_1}{dx}\psi_1(x)\,dx\right\} \\
&= i\hbar\int_0^\infty \psi_1(x)\frac{d\psi_1}{dx}\,dx \\
&= 0.
\end{aligned}$$

The expectation value of p^2 at time t is

$$\begin{aligned}
\langle p^2\rangle = \int_0^\infty \Psi_1^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi_1(x,t)\,dx &= -\hbar^2\int_0^\infty \psi_1(x)\frac{d^2\psi_1}{dx^2}\,dx \\
&= -\hbar^2\left\{\underbrace{\psi_1(x)\frac{d\psi_1}{dx}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{d\psi_1}{dx}\frac{d\psi_1}{dx}\,dx\right\} \\
&= \hbar^2\int_0^\infty \left(\frac{d\psi_1}{dx}\right)^2 dx \\
&\approx \hbar^2\int_0^\infty \left[\frac{1.426}{\sqrt{x_0}}\frac{d}{dx}\,\text{Ai}\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\
&\approx \hbar^2\int_0^\infty \left[\frac{1.426}{\sqrt{x_0}}\frac{1}{x_0}\,\text{Ai}'\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\
&\approx \frac{2.033\hbar^2}{x_0^3}\int_0^\infty \left[\text{Ai}'\left(\frac{x}{x_0}-\epsilon_1\right)\right]^2 dx \\
&\approx \frac{2.033\hbar^2}{x_0^3}\int_0^\infty [\text{Ai}'(z-\epsilon_1)]^2(x_0\,dz) \\
&\approx \frac{2.033\hbar^2}{x_0^2}\int_0^\infty [\text{Ai}'(z-\epsilon_1)]^2\,dz \\
&\approx \frac{2.033\hbar^2}{x_0^2}\int_{-\epsilon_1}^\infty [\text{Ai}'(u)]^2\,du \\
&\approx \frac{0.7791\hbar^2}{x_0^2}
\end{aligned}$$

The standard deviation in p for the first stationary state is then

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \approx \sqrt{\frac{0.7791\hbar^2}{x_0^2}} \approx \frac{0.8827\hbar}{x_0}.$$

Consequently, the uncertainty product for the first stationary state is

$$\sigma_x\sigma_p \approx (0.6973x_0)\left(\frac{0.8827\hbar}{x_0}\right) \approx 0.6155\hbar,$$

which is consistent with Heisenberg's principle ($\sigma_x\sigma_p \geq 0.5\hbar$). Now calculate the expectation values for the tenth stationary state. The expectation value of x at time t is

$$\begin{aligned}
\langle x\rangle = \int_0^\infty \Psi_{10}^*(x,t)(x)\Psi_{10}(x,t)\,dx = \int_0^\infty x[\psi_{10}(x)]^2\,dx &\approx \int_0^\infty x\left[\frac{0.9365}{\sqrt{x_0}}\operatorname{Ai}\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770}{x_0}\int_0^\infty x\left[\operatorname{Ai}\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770}{x_0}\int_0^\infty x_0 z\,[\operatorname{Ai}(z-\epsilon_{10})]^2\,(x_0\,dz) \\
&\approx 0.8770x_0\int_0^\infty z\,[\operatorname{Ai}(z-\epsilon_{10})]^2\,dz \\
&\approx 0.8770x_0\int_{-\epsilon_{10}}^\infty (u+\epsilon_{10})\,[\operatorname{Ai}(u)]^2\,du \\
&\approx 8.554x_0.
\end{aligned}$$

The expectation value of x^2 at time t is

$$\begin{aligned}
\langle x^2\rangle = \int_0^\infty \Psi_{10}^*(x,t)(x^2)\Psi_{10}(x,t)\,dx = \int_0^\infty x^2[\psi_{10}(x)]^2\,dx &\approx \int_0^\infty x\left[\frac{0.9365}{\sqrt{x_0}}\operatorname{Ai}\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770}{x_0}\int_0^\infty x^2\left[\operatorname{Ai}\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770}{x_0}\int_0^\infty x_0^2 z^2\,[\operatorname{Ai}(z-\epsilon_{10})]^2\,(x_0\,dz) \\
&\approx 0.8770x_0^2\int_0^\infty z^2\,[\operatorname{Ai}(z-\epsilon_{10})]^2\,dz \\
&\approx 0.8770x_0^2\int_{-\epsilon_{10}}^\infty (u+\epsilon_{10})^2\,[\operatorname{Ai}(u)]^2\,du \\
&\approx 87.79x_0^2.
\end{aligned}$$

The standard deviation in x for the tenth stationary state is then

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \approx \sqrt{87.79x_0^2 - 8.554^2x_0^2} \approx 3.824x_0.$$

The expectation value of p at time t is

$$\begin{aligned}
\langle p\rangle = \int_0^\infty \Psi_{10}^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi_{10}(x,t)\,dx &= -i\hbar\int_0^\infty \psi_{10}(x)\frac{d\psi_{10}}{dx}\,dx \\
&= -i\hbar\left\{\underbrace{[\psi_{10}(x)]^2\Big|_0^\infty}_{=\,0} - \int_0^\infty \frac{d\psi_{10}}{dx}\psi_{10}(x)\,dx\right\} \\
&= i\hbar\int_0^\infty \psi_{10}(x)\frac{d\psi_{10}}{dx}\,dx \\
&= 0.
\end{aligned}$$

The expectation value of p^2 at time t is

$$\begin{aligned}
\langle p^2\rangle = \int_0^\infty \Psi_{10}^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi_{10}(x,t)\,dx &= -\hbar^2\int_0^\infty \psi_{10}(x)\frac{d^2\psi_{10}}{dx^2}\,dx \\
&= -\hbar^2\left\{\underbrace{\psi_{10}(x)\frac{d\psi_{10}}{dx}\Big|_0^\infty}_{=\,0} - \int_0^\infty \frac{d\psi_{10}}{dx}\frac{d\psi_{10}}{dx}\,dx\right\} \\
&= \hbar^2\int_0^\infty \left(\frac{d\psi_{10}}{dx}\right)^2 dx \\
&\approx \hbar^2\int_0^\infty \left[\frac{0.9365}{\sqrt{x_0}}\frac{d}{dx}\,\mathrm{Ai}\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \hbar^2\int_0^\infty \left[\frac{0.9365}{\sqrt{x_0}}\frac{1}{x_0}\,\mathrm{Ai}'\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770\hbar^2}{x_0^3}\int_0^\infty \left[\mathrm{Ai}'\left(\frac{x}{x_0}-\epsilon_{10}\right)\right]^2 dx \\
&\approx \frac{0.8770\hbar^2}{x_0^3}\int_0^\infty [\mathrm{Ai}'(z-\epsilon_{10})]^2(x_0\,dz) \\
&\approx \frac{0.8770\hbar^2}{x_0^2}\int_0^\infty [\mathrm{Ai}'(z-\epsilon_{10})]^2\,dz \\
&\approx \frac{0.8770\hbar^2}{x_0^2}\int_{-\epsilon_{10}}^\infty [\mathrm{Ai}'(u)]^2\,du \\
&\approx \frac{4.277\hbar^2}{x_0^2}
\end{aligned}$$

The standard deviation in p for the tenth stationary state is then

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \approx \sqrt{\frac{4.277\hbar^2}{x_0^2}} \approx \frac{2.068\hbar}{x_0}.$$

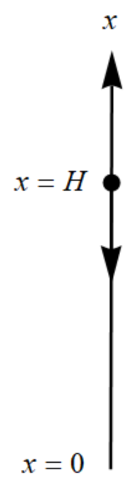
Consequently, the uncertainty product for the tenth stationary state is

$$\sigma_x\sigma_p \approx (3.824x_0)\left(\frac{2.068\hbar}{x_0}\right) \approx 7.909\hbar,$$

which is consistent with Heisenberg's principle ($\sigma_x\sigma_p \geq 0.5\hbar$).

Part (c)

Suppose there's a mass m held at a height $x = H$ above the ground and then released.



The total mechanical energy E is the sum of the potential and kinetic energies, assuming there are no resistive forces.

$$E = mgx + \frac{1}{2}mv^2$$

Solve for the velocity v.

$$v = \pm\sqrt{\frac{2}{m}(E - mgx)}$$

From Problem 1.11 on page 20, the classical probability distribution is given by

$$\rho_C(x)\,dx = \frac{1}{v(x)T}\,dx,$$

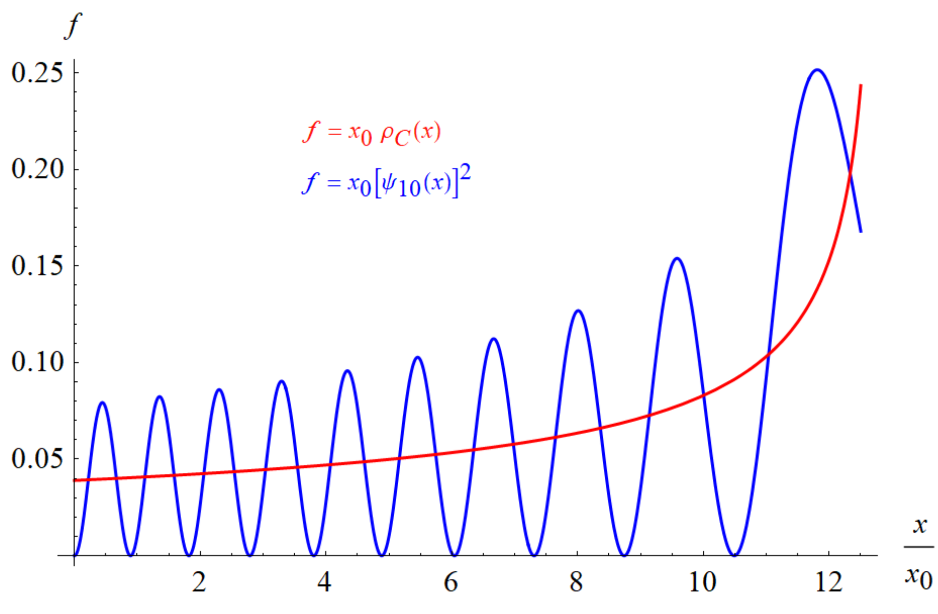
where T is the amount of time it takes to go from start to finish.

$$\begin{aligned}\rho_C(x) = \frac{1}{v(x)\displaystyle\int_a^b \frac{1}{v(x)}\,dx} = \frac{1}{\sqrt{\frac{2}{m}(E-mgx)}\displaystyle\int_H^0 \frac{1}{\sqrt{\frac{2}{m}(E-mgx)}}(-dx)} &= \frac{1}{\sqrt{E-mgx}\displaystyle\int_0^H (E-mgx)^{-1/2}\,dx} \\ &= \frac{1}{\sqrt{E-mgx}\cdot\frac{2}{mg}(\sqrt{E}-\underbrace{\sqrt{E-mgH}}_{=0})} \\ &= \frac{mg}{2\sqrt{E(E-mgx)}}\end{aligned}$$

Now write $\rho_C(x)$ in terms of the dimensionless energy.

$$\begin{aligned}
\rho_C(x) = \frac{mg}{2\sqrt{E(E-mgx)}} = \frac{1}{\frac{2}{mg}\sqrt{E(E-mgx)}} &= \frac{1}{\frac{2x_0}{mgx_0}\sqrt{E(E-mgx)}} \\
&= \frac{1}{2x_0\sqrt{\frac{E}{m^2g^2x_0^2}(E-mgx)}} \\
&= \frac{1}{2x_0\sqrt{\frac{E}{mgx_0}\left(\frac{E}{mgx_0}-\frac{x}{x_0}\right)}} \\
&= \frac{1}{2x_0\sqrt{\epsilon\left(\epsilon-\frac{x}{x_0}\right)}}
\end{aligned}$$

Below is a plot of $x_0\rho_C(x)$ with $\epsilon = \epsilon_{10}$ and $x_0[\psi_{10}(x)]^2$ versus x/x_0.



The classical probability distribution looks like an averaged-out version of the quantum one. While the particle is likely to be found only within certain intervals in the tenth eigenstate, the particle is likely to be found anywhere in the classical case.

Part (d)

One final note: The nondimensionalization procedure used in part (a) can be applied to PDEs as well as ODEs. For example, consider the Schrödinger equation from earlier.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + mgx\Psi(x,t), \quad x > 0,\ t > 0$$

Seeing as how all terms have units of energy, divide both sides by mgx_0.

$$i\frac{\hbar}{mgx_0}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m^2gx_0}\frac{\partial^2\Psi}{\partial x^2} + \frac{x}{x_0}\Psi(x,t)$$

Introduce the dimensionless position variable, $z = x/x_0$.

$$i\frac{\hbar}{mgx_0}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m^2gx_0^3}\frac{\partial^2\Psi}{\partial z^2} + z\Psi(z,t)$$

Choose x_0 so that the coefficient of the second derivative vanishes.

$$i\frac{\hbar}{mgx_0}\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial z^2} + z\Psi(z,t)$$

Then introduce the dimensionless time variable, $s = (mgx_0/\hbar)t$.

$$i\frac{\partial\Psi}{\partial s} = -\frac{\partial^2\Psi}{\partial z^2} + z\Psi(z,s), \quad z > 0,\ s > 0$$

This is the purest expression of Schrödinger's equation because it's not cluttered with constants.