Problem 2.60

The $1/x^2$ potential. Suppose

$$V(x) = \begin{cases} -\alpha/x^2, & x > 0, \\ \infty, & x \leq 0. \end{cases} \tag{2.189}$$

where α is some positive constant with the appropriate dimensions. We'd like to find the bound states—solutions to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\alpha}{x^2}\psi = E\psi \tag{2.190}$$

with negative energy ($E < 0$).

(a) Let's first go for the ground state energy, E_0. Prove, on dimensional grounds, that there is *no possible formula* for E_0—no way to construct (from the available constants m, $\hbar$, and α) a quantity with the units of energy. That's weird, but it gets worse. . . .

(b) For convenience, rewrite Equation 2.190 as

$$\frac{d^2\psi}{dx^2} + \frac{\beta}{x^2}\psi = \kappa^2\psi, \quad \text{where } \beta \equiv \frac{2m\alpha}{\hbar^2} \text{ and } \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}. \tag{2.191}$$

Show that if $\psi(x)$ satisfies this equation with energy E, then so too does $\psi(\lambda x)$, with energy $E' = \lambda^2 E$, for any positive number λ. [This is a catastrophe: if there exists any solution at all, then there's a solution for *every* (negative) energy! Unlike the square well, the harmonic oscillator, and every other potential well we have encountered, there are no discrete allowed values—and no ground state. A system with no ground state—no lowest allowed energy—would be wildly unstable, cascading down to lower and lower levels, giving off an unlimited amount of energy as it falls. It might solve our energy problem, but we'd all be fried in the process.] Well, perhaps there simply are *no solutions at all*. . . .

(c) (Use a computer for the remainder of this problem.) Show that

$$\psi_\kappa(x) = A\sqrt{x}K_{ig}(\kappa x), \tag{2.192}$$

satisfies Equation 2.191 (here K_{ig} is the modified Bessel function of order ig, and $g \equiv \sqrt{\beta - 1/4}$). Plot this function, for $g = 4$ (you might as well let $\kappa = 1$ for the graph; this just sets the scale of length). Notice that it goes to 0 as $x \to 0$ and as $x \to \infty$. And it's normalizable: determine A.[66] How about the old rule that the number of nodes counts the number of lower-energy states? This function has an *infinite* number of nodes, regardless of the energy (i.e. of κ). I guess that's consistent, since for any E there are always an infinite number of states with even lower energy.

(d) This potential confounds practically everything we have come to expect. The problem is that it blows up too violently as $x \to 0$.

[66] $\psi_\kappa(x)$ is normalizable as long as g is real—which is to say, provided $\beta > 1/4$. For more on this strange problem see A. M. Essin and D. J. Griffiths, *Am. J. Phys.* **74**, 109 (2006), and references therein.

If you move the "brick wall" over a hair,

$$V(x) = \begin{cases} -\alpha/x^2, & x > \epsilon > 0, \\ \infty, & x \leq \epsilon, \end{cases} \tag{2.193}$$

it's suddenly perfectly normal. Plot the ground state wave function, for $g = 4$ and $\epsilon = 1$ (you'll first need to determine the appropriate value of κ), from $x = 0$ to $x = 6$. Notice that we have introduced a new parameter (ϵ), with the dimensions of length, so the argument in (a) is out the window. Show that the ground state energy takes the form

$$E_0 = -\frac{\alpha}{\epsilon^2} f(\beta), \tag{2.194}$$

for some function f of the dimensionless quantity β.

TYPO: The period in Equation 2.189 should be a comma. To be more accuate, the independent clause should read, "here K_{ig} is the modified Bessel function of the second kind of order ig."

Solution

Part (a)

Schrödinger's equation governs the time evolution of the wave function.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty, \ t > 0$$

There are two boundary conditions as $|x|$ becomes large: $\Psi \to 0$ as $x \to \pm\infty$. For this problem, the potential energy function is

$$V(x,t) = V(x) = \begin{cases} -\dfrac{\alpha}{x^2} & \text{if } x > 0 \\ \infty & \text{if } x \leq 0 \end{cases}.$$

Note that because x has SI units of meters and $V(x)$ has SI units of joules, α has units of joule $\cdot$ meters^2. The units of each of the constants are as follows.

$$[m] = \text{kg}$$

$$[\hbar] = \text{J} \cdot \text{s} = \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \cdot \text{s} = \frac{\text{kg} \cdot \text{m}^2}{\text{s}}$$

$$[\alpha] = \text{J} \cdot \text{m}^2 = \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \cdot \text{m}^2 = \frac{\text{kg} \cdot \text{m}^4}{\text{s}^2}$$

Combine the constants in an arbitrary fashion.

$$[m]^a [\hbar]^b [\alpha]^c = (\text{kg})^a \left(\frac{\text{kg} \cdot \text{m}^2}{\text{s}}\right)^b \left(\frac{\text{kg} \cdot \text{m}^4}{\text{s}^2}\right)^c = \text{kg}^{a+b+c} \text{m}^{2b+4c} \text{s}^{-b-2c}$$

In order for this combination to be in joules, the following system of equations must be satisfied.

$$a + b + c = 1$$
$$2b + 4c = 2$$
$$-b - 2c = -2$$

Multiplying both sides of this third equation by -2 gives $2b + 4c = 4$. It's impossible to choose b and c such that $2b + 4c = 2$ and $2b + 4c = 4$ because $2 \neq 4$. Therefore, no combination of the available constants results in a quantity with units of energy, and no formula for the ground state energy exists as a result.

Part (b)

Split up the Schrödinger equation over the intervals that $V(x)$ is defined on.

$$\begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + (\infty)\Psi(x,t), & x \leq 0 \\ i\hbar \dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} - \dfrac{\alpha}{x^2}\Psi(x,t), & x > 0 \end{cases}$$

The only way the first PDE can be satisfied is if $\Psi(x,t) = 0$ for $x \leq 0$. Since the wave function is continuous, the boundary condition $\Psi(0,t) = 0$ becomes associated with the PDE over $x > 0$.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\alpha}{x^2}\Psi(x,t), \quad x > 0, \ t > 0$$
$$\Psi(0,t) = 0$$
$$\Psi(\infty,t) = 0$$

Since Schrödinger's equation and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution for $\Psi(x,t) = \phi(t)\psi(x)$ and plug it into the PDE

$$i\hbar\phi'(t)\psi(x) = -\frac{\hbar^2}{2m}\phi(t)\psi''(x) - \frac{\alpha}{x^2}\phi(t)\psi(x)$$

and the boundary conditions.

$$\Psi(0,t) = 0 \quad \rightarrow \quad \phi(t)\psi(0) = 0 \quad \rightarrow \quad \psi(0) = 0$$
$$\Psi(\infty,t) = 0 \quad \rightarrow \quad \phi(t)\psi(\infty) = 0 \quad \rightarrow \quad \psi(\infty) = 0$$

Divide both sides of the PDE by $\phi(t)\psi(x)$ to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\alpha}{x^2}$$

The only way a function of t can be equal to a function of x is if both are equal to the same constant.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\alpha}{x^2} = E$$

By applying the method of separation of variables, the PDE has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\alpha}{x^2} &= E \end{aligned}\right\}$$

The ODE in x is known as the time-independent Schrödinger equation (TISE).

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\alpha}{x^2}\psi(x) = E\psi(x)$$

Multiply both sides by $-2m/\hbar^2$.

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2 x^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x) \tag{1}$$

Suppose that $\psi(x)$ is a known solution and that E is its corresponding energy. Check to see if $\psi(\lambda x)$ with corresponding energy E' also satisfies it.

$$\frac{d^2\psi(\lambda x)}{dx^2} + \frac{2m\alpha}{\hbar^2 x^2}\psi(\lambda x) = -\frac{2mE'}{\hbar^2}\psi(\lambda x)$$

Make the change of variables $r = \lambda x$.

$$\frac{d^2\psi(r)}{dx^2} + \frac{2m\alpha}{\hbar^2 \left(\frac{r}{\lambda}\right)^2}\psi(r) = -\frac{2mE'}{\hbar^2}\psi(r)$$

Use the chain rule to write the second derivative in terms of this new variable.

$$\frac{d\psi}{dx} = \frac{d\psi}{dr}\frac{dr}{dx} = \frac{d\psi}{dr}(\lambda)$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{dr}{dx}\frac{d}{dr}\left(\lambda\frac{d\psi}{dr}\right) = \lambda^2\frac{d^2\psi}{dr^2}$$

Consequently,

$$\lambda^2\frac{d^2\psi}{dr^2} + \frac{2m\alpha\lambda^2}{\hbar^2 r^2}\psi(r) = -\frac{2mE'}{\hbar^2}\psi(r).$$

Divide both sides by λ^2.

$$\frac{d^2\psi}{dr^2} + \frac{2m\alpha}{\hbar^2 r^2}\psi(r) = -\frac{2mE'}{\hbar^2\lambda^2}\psi(r) \tag{2}$$

This is the same equation that $\psi(x)$ satisfies [equation (1)] but with r instead.

$$\frac{d^2\psi}{dr^2} + \frac{2m\alpha}{\hbar^2 r^2}\psi(r) = -\frac{2mE'}{\hbar^2\lambda^2}\psi(r) = -\frac{2mE}{\hbar^2}\psi(r)$$

Therefore, if $\psi(x)$ is a solution with energy E, then $\psi(\lambda x)$ is also a solution with energy

$$\frac{E'}{\lambda^2} = E \quad \rightarrow \quad E' = \lambda^2 E.$$

Part (c)

Repeat equation (1).

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2 x^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

Let $\beta = 2m\alpha/\hbar^2$ and $\kappa^2 = -2mE/\hbar^2$.

$$\frac{d^2\psi}{dx^2} + \frac{\beta}{x^2}\psi(x) = \kappa^2\psi(x)$$

Make the inspired substitution $\psi(x) = \sqrt{x}\,\zeta(x)$.

$$\frac{d^2}{dx^2}[\sqrt{x}\,\zeta(x)] + \frac{\beta}{x^2}[\sqrt{x}\,\zeta(x)] = \kappa^2[\sqrt{x}\,\zeta(x)]$$

Evaluate the derivative.

$$\left[x^{1/2}\zeta''(x) + x^{-1/2}\zeta'(x) - \frac{1}{4}x^{-3/2}\zeta(x)\right] + \beta x^{-3/2}\zeta(x) = \kappa^2 x^{1/2}\zeta(x)$$

Combine like-terms on the left side.

$$x^{1/2}\zeta''(x) + x^{-1/2}\zeta'(x) + \frac{\beta - \frac{1}{4} - \kappa^2x^2}{x^{3/2}}\zeta(x) = 0$$

Multiply both sides by $x^{3/2}$.

$$x^2\zeta''(x) + x\zeta'(x) + \left(\beta - \frac{1}{4} - \kappa^2x^2\right)\zeta(x) = 0$$

Let $g = \sqrt{\beta - 1/4}$.

$$x^2\zeta''(x) + x\zeta'(x) + (g^2 - \kappa^2x^2)\zeta(x) = 0$$

Make another change of variables $z = \kappa x$.

$$\frac{z^2}{\kappa^2}\frac{d^2\zeta}{dx^2} + \frac{z}{\kappa}\frac{d\zeta}{dx} + (g^2 - z^2)\zeta(z) = 0$$

Use the chain rule again to find the derivatives in terms of this new variable.

$$\frac{d\zeta}{dx} = \frac{d\zeta}{dz}\frac{dz}{dx} = \frac{d\zeta}{dz}(\kappa)$$

$$\frac{d^2\zeta}{dx^2} = \frac{d}{dx}\left(\frac{d\zeta}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left(\kappa\frac{d\zeta}{dz}\right) = \kappa^2\frac{d^2\zeta}{dz^2}$$

Consequently,

$$\frac{z^2}{\kappa^2}\left(\kappa^2\frac{d^2\zeta}{dz^2}\right) + \frac{z}{\kappa}\left(\kappa\frac{d\zeta}{dz}\right) + (g^2 - z^2)\zeta(z) = 0.$$

Simplify the left side.

$$z^2\frac{d^2\zeta}{dz^2} + z\frac{d\zeta}{dz} - [z^2 + (ig)^2]\zeta(z) = 0$$

This is the modified Bessel equation of order ig. Its general solution can be written in terms of I and K, the modified Bessel functions of the first and second kind, respectively.

$$\zeta(z) = C_1 I_{ig}(z) + C_2 K_{ig}(z)$$

Now that the solution is known, change back to the original variables.

$$\zeta(x) = C_1 I_{ig}(\kappa x) + C_2 K_{ig}(\kappa x)$$

Since $\psi(x) = \sqrt{x}\,\zeta(x)$, the general solution is

$$\psi(x) = \sqrt{x}\left[C_1 I_{ig}(\kappa x) + C_2 K_{ig}(\kappa x)\right].$$

In order to satisfy the boundary conditions, $\psi(0) = 0$ and $\psi(\infty) = 0$, set $C_1 = 0$.

$$\psi(x) = C_2\sqrt{x}\,K_{ig}(\kappa x)$$

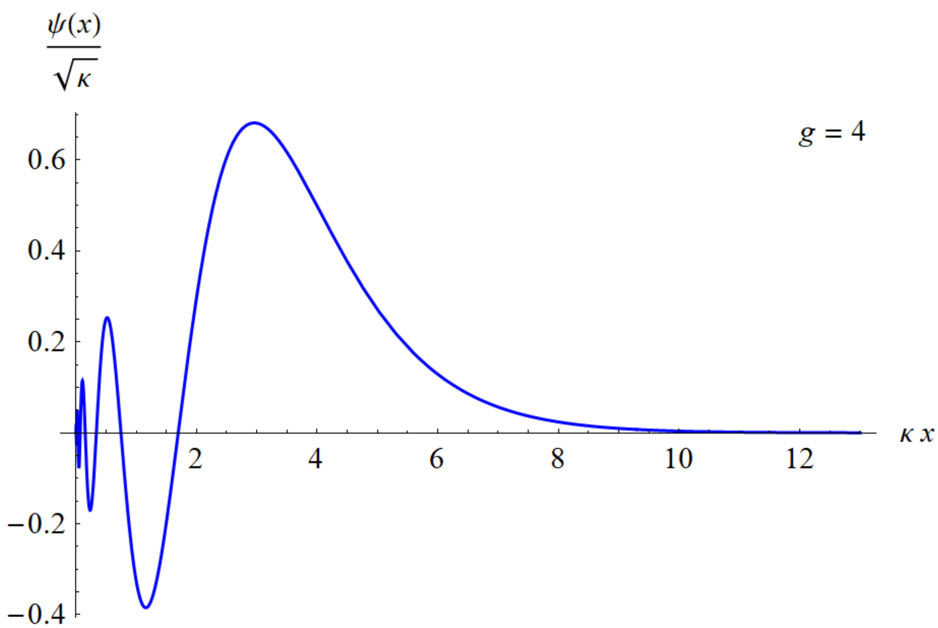
Finally, normalize $\psi(x)$ to determine C_2.

$$1 = \int_0^\infty [\psi(x)]^2\,dx = \int_0^\infty \{C_2^2 x[K_{ig}(\kappa x)]^2\}\,dx = \frac{C_2^2\pi g}{2\kappa^2\sinh(\pi g)} \quad \rightarrow \quad C_2 = \pm\kappa\sqrt{\frac{2\sinh(\pi g)}{\pi g}}$$

Therefore,

$$\boxed{\psi(x) = \kappa\sqrt{\frac{2\sinh(\pi g)}{\pi g}}\sqrt{x}\,K_{ig}(\kappa x).}$$

A plot of the normalized eigenstate versus κx is shown below for the special case that $g = 4$.



Part (d)

Suppose the potential energy function is now

$$V(x) = \begin{cases} -\dfrac{\alpha}{x^2} & \text{if } x > \epsilon > 0 \\ \infty & \text{if } x \leq \epsilon \end{cases}.$$

Because x has SI units of meters and $V(x)$ has SI units of joules, α has units of joule · meters^2. The units of each of the constants are as follows.

$$\begin{aligned}
[m] &= \text{kg} \\
[\hbar] &= \text{J}\cdot\text{s} = \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}\cdot\text{s} = \frac{\text{kg}\cdot\text{m}^2}{\text{s}} \\
[\alpha] &= \text{J}\cdot\text{m}^2 = \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}\cdot\text{m}^2 = \frac{\text{kg}\cdot\text{m}^4}{\text{s}^2} \\
[\epsilon] &= \text{m}
\end{aligned}$$

Combine the constants in an arbitrary fashion.

$$[m]^a[\hbar]^b[\alpha]^c[\epsilon]^d = (\text{kg})^a\left(\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\right)^b\left(\frac{\text{kg}\cdot\text{m}^4}{\text{s}^2}\right)^c(\text{m})^d = \text{kg}^{a+b+c}\text{m}^{2b+4c+d}\text{s}^{-b-2c}$$

In order for this combination to be in joules, the following system of equations must be satisfied.

$$\begin{aligned}
a+b+c &= 1 \\
2b+4c+d &= 2 \\
-b-2c &= -2
\end{aligned}$$

Solving this system yields $b = -2a$, $c = 1+a$, and $d = -2$, where a is a free variable. For the simplest combination, choose $a = 0$.

$$m^0\hbar^0\alpha^1\epsilon^{-2} = \frac{\alpha}{\epsilon^2}.$$

In order for the combination to be dimensionless, look for nontrivial solutions of

$$\begin{aligned}
a+b+c &= 0 \\
2b+4c+d &= 0 \\
-b-2c &= 0.
\end{aligned}$$

Solving it yields $a = c$, $b = -2c$, and $d = 0$, where c is a free variable. For the simplest one, choose $c = 1$.

$$m^1\hbar^{-2}\alpha^1\epsilon^0 = \frac{m\alpha}{\hbar^2}$$

Therefore, based on an analysis of the dimensions, the ground state energy has the form,

$$E_0 = \frac{\alpha}{\epsilon^2}F\left(\frac{m\alpha}{\hbar^2}\right) = -\frac{\alpha}{\epsilon^2}f\left(\frac{2m\alpha}{\hbar^2}\right) = -\frac{\alpha}{\epsilon^2}f(\beta),$$

where F and f are arbitrary functions.