Problem 2.64

Legendre's differential equation reads

$$\left(1-x^2\right)\frac{d^2f}{dx^2}-2x\frac{df}{dx}+\ell(\ell+1)f=0, \tag{2.208}$$

where ℓ is some (non-negative) real number.

(a) Assume a power series solution,

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

and obtain a recursion relation for the constants a_n.

(b) Argue that unless the series truncates (which can only happen if ℓ is an integer), the solution will diverge at $x=1$.

(c) When ℓ *is* an integer, the series for one of the two linearly independent solutions (either f_{even} or f_{odd} depending on whether ℓ is even or odd) will truncate, and those solutions are called **Legendre polynomials** $P_\ell(x)$. Find $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ from the recursion relation. Leave your answer in terms of either a_0 or a_1.[69]

Solution

Part (a)

$x=0$ is an ordinary point, so there is a series solution for f about this point.

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

The zeros of $1-x^2$ are ±1, which means the series solution will have a convergence radius of at least 1 ($-1<x<1$). Differentiate it to obtain formulas for df/dx and d^2f/dx^2.

$$f(x)=\sum_{n=0}^{\infty}a_nx^n \quad\rightarrow\quad \frac{df}{dx}=\sum_{n=1}^{\infty}na_nx^{n-1} \quad\rightarrow\quad \frac{d^2f}{dx^2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

Substitute them into the ODE to determine a_n.

$$\left(1-x^2\right)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-2x\sum_{n=1}^{\infty}na_nx^{n-1}+\ell(\ell+1)\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-x^2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-2x\sum_{n=1}^{\infty}na_nx^{n-1}+\ell(\ell+1)\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=1}^{\infty}2na_nx^n+\ell(\ell+1)\sum_{n=0}^{\infty}a_nx^n=0$$

[69]By convention Legendre polynomials are normalized such that $P_\ell(1)=1$. Note that the nonvanishing coefficients will take different values for different ℓ.

Substitute $n = k + 2$ in the first sum and $n = k$ in the second, third, and fourth sums.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} 2ka_k x^k + \ell(\ell+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

Start the second and third sums from $k = 0$.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 2ka_k x^k + \sum_{k=0}^{\infty} \ell(\ell+1)a_k x^k = 0$$

Combine the sums.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2}x^k - k(k-1)a_k x^k - 2ka_k x^k + \ell(\ell+1)a_k x^k\right] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + \ell(\ell+1)a_k\right] x^k = 0$$

The quantity in square brackets must be zero.

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + \ell(\ell+1)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k(k-1) + 2k - \ell(\ell+1)}{(k+2)(k+1)} a_k$$

Therefore, changing back to n,

$$a_{n+2} = \frac{(n^2 - \ell^2) + (n - \ell)}{(n+2)(n+1)} a_n.$$

The constants, a_0 and a_1, are arbitrary, and all subsequent constants (a_2, a_3, …) are determined through this recursion relation.

Part (b)

The series solution is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Setting $x = 1$,

$$f(1) = \sum_{n=0}^{\infty} a_n.$$

This is an infinite series that diverges generally because a_0 and a_1 are arbitrary. It can only converge if the series becomes finite somehow—either if $a_0 = a_1 = 0$ (the trivial solution) or if ℓ is an integer. If ℓ is an odd integer, then a_ℓ, $a_{\ell-2}$, …, a_1 will be nonzero and $a_{\ell+2}$, $a_{\ell+4}$, … will be zero; set $a_0 = 0$ in this case to make the series finite. If ℓ is an even integer, then a_ℓ, $a_{\ell-2}$, …, a_0 will be nonzero and $a_{\ell+2}$, $a_{\ell+4}$, … will be zero; set $a_1 = 0$ in this case to make the series finite.

Part (c)

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$\ell = 0:$ a_0 is arbitrary; $a_2,\ a_4, \ldots$ are zero.
Set $a_1 = 0$ to make $a_3 = a_5 = \cdots = 0$ so that the series becomes finite.
$f(x) = a_0$

Therefore, $P_0(x) = a_0$.

$\ell = 1:$ a_1 is arbitrary; $a_3,\ a_5, \ldots$ are zero.
Set $a_0 = 0$ to make $a_2 = a_4 = \cdots = 0$ so that the series becomes finite.
$f(x) = a_1 x$

Therefore, $P_1(x) = a_1 x$.

$\ell = 2:$ a_0 is arbitrary; $a_4,\ a_6, \ldots$ are zero.
Set $a_1 = 0$ to make $a_3 = a_5 = \cdots = 0$ so that the series becomes finite.

$$f(x) = a_0 + a_2 x^2 = a_0 + \left[\frac{(0^2 - 2^2) + (0 - 2)}{(0+2)(0+1)} a_0\right] x^2 = a_0 - 3a_0 x^2 = a_0(1 - 3x^2)$$

Therefore, $P_2(x) = a_0(1 - 3x^2)$.

$\ell = 3:$ a_1 is arbitrary; $a_5,\ a_7, \ldots$ are zero.
Set $a_0 = 0$ to make $a_2 = a_4 = \cdots = 0$ so that the series becomes finite.

$$f(x) = a_1 x + a_3 x^3 = a_1 x + \left[\frac{(1^2 - 3^2) + (1 - 3)}{(1+2)(1+1)} a_1\right] x^3 = a_1 x - \frac{5}{3} a_1 x^3 = a_1\left(x - \frac{5}{3} x^3\right)$$

Therefore, $P_3(x) = a_1\left(x - \frac{5}{3} x^3\right)$.