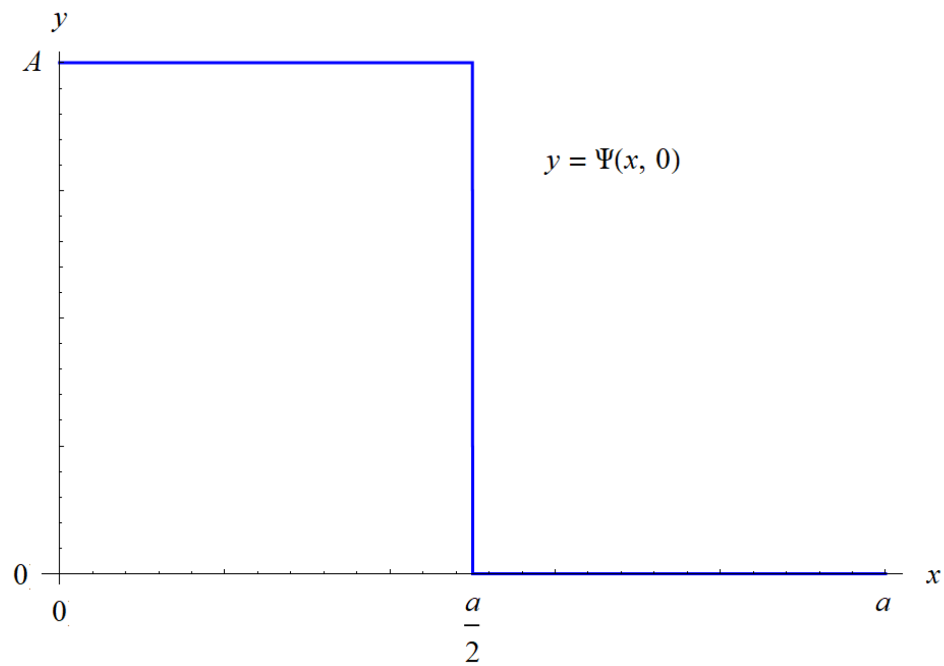
Problem 2.8

A particle of mass m in the infinite square well (of width a) starts out in the state

$$\Psi(x,0)=\begin{cases}A, & 0\le x\le a/2,\\ 0, & a/2\le x\le a,\end{cases}$$

for some constant A, so it is (at $t=0$) equally likely to be found at any point in the left half of the well. What is the probability that a measurement of the energy (at some later time t) would yield the value $\pi^2\hbar^2/2ma^2$?

Solution



In Problem 2.3 the general solution to the Schrödinger equation for the infinite square well potential,

$$V(x)=\begin{cases}0 & \text{if } 0\le x\le a\\ \infty & \text{otherwise}\end{cases},$$

was found to be

$$\Psi(x,t)=\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}B_n\exp\left(-i\frac{\hbar\pi^2n^2}{2ma^2}t\right)\sin\frac{n\pi x}{a},\quad 0\le x\le a$$

and zero elsewhere. The coefficients B_n are determined by using the provided initial condition.

Before doing so, though, first normalize the initial wave function by finding A.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\ &= \int_{-\infty}^{\infty} \Psi(x,0)\Psi^*(x,0)\, dx \\ &= \int_0^{a/2} (A)(A)^* \, dx + \int_{a/2}^{a} (0)(0)^* \, dx \\ &= A^2 \int_0^{a/2} dx \\ &= A^2 \left(\frac{a}{2}\right) \end{aligned}$$

Solve for A.

$$A = \sqrt{\frac{2}{a}}$$

As a result, the initial wave function becomes

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{a}}, & 0 \le x \le a/2, \\ 0, & a/2 \le x \le a, \end{cases}.$$

Set $t = 0$ in the general solution.

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}$$

To solve for B_n, multiply both sides by $\sin\frac{p\pi x}{a}$, where p is an integer,

$$\Psi(x,0)\sin\frac{p\pi x}{a} = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sin\frac{p\pi x}{a}$$

and then integrate both sides with respect to x from 0 to a.

$$\begin{aligned} \int_0^a \Psi(x,0)\sin\frac{p\pi x}{a}\, dx &= \int_0^a \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sin\frac{p\pi x}{a}\, dx \\ &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \int_0^a \sin\frac{n\pi x}{a} \sin\frac{p\pi x}{a}\, dx \end{aligned}$$

Because the sine functions are orthogonal, this integral on the right is zero if $n \neq p$. Every term in this infinite series vanishes, then, except for the one corresponding to $n = p$.

$$\begin{aligned} \int_0^a \Psi(x,0)\sin\frac{n\pi x}{a}\, dx &= \sqrt{\frac{2}{a}} B_n \int_0^a \sin^2\frac{n\pi x}{a}\, dx \\ &= \sqrt{\frac{2}{a}} B_n \left(\frac{a}{2}\right) \\ &= \sqrt{\frac{a}{2}} B_n \end{aligned}$$

Solve for B_n.

$$\begin{aligned}
B_n &= \sqrt{\frac{2}{a}} \int_0^a \Psi(x,0) \sin\frac{n\pi x}{a}\,dx \\
&= \sqrt{\frac{2}{a}} \left(\int_0^{a/2} \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}\,dx + \int_{a/2}^a 0 \sin\frac{n\pi x}{a}\,dx \right) \\
&= \frac{2}{a} \int_0^{a/2} \sin\frac{n\pi x}{a}\,dx \\
&= \frac{2}{a} \left(-\frac{a}{n\pi} \cos\frac{n\pi x}{a} \right) \Big|_0^{a/2} \\
&= \frac{2}{a} \left(-\frac{a}{n\pi} \cos\frac{n\pi}{2} + \frac{a}{n\pi} \right) \\
&= \frac{2}{n\pi} \left(1 - \cos\frac{n\pi}{2} \right)
\end{aligned}$$

With these coefficients, the general solution becomes

$$\begin{aligned}
\Psi(x,t) &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} B_n \exp\left(-i\frac{\hbar\pi^2 n^2}{2ma^2} t \right) \sin\frac{n\pi x}{a} \\
&= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - \cos\frac{n\pi}{2} \right) \exp\left(-i\frac{\hbar\pi^2 n^2}{2ma^2} t \right) \sin\frac{n\pi x}{a}, \quad 0 \le x \le a.
\end{aligned}$$

Therefore,

$$\boxed{\Psi(x,t) = \frac{2}{\pi} \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\frac{n\pi}{2} \right) \exp\left(-i\frac{\hbar\pi^2 n^2}{2ma^2} t \right) \sin\frac{n\pi x}{a}, \quad 0 \le x \le a.}$$

Writing the general solution in terms of the eigenstates,

$$\begin{aligned}
\Psi(x,t) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\frac{n\pi}{2} \right) \left(\sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \right) \exp\left(-i\frac{\hbar\pi^2 n^2}{2ma^2} t \right) \\
&= \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - \cos\frac{n\pi}{2} \right) \psi_n(x) \exp\left(-i\frac{\hbar\pi^2 n^2}{2ma^2} t \right),
\end{aligned}$$

we can see that the probability of measuring energy,

$$E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2},$$

is

$$P(E_n) = \left| \frac{2}{n\pi} \left(1 - \cos\frac{n\pi}{2} \right) \right|^2 = \frac{4}{n^2\pi^2} \left(1 - \cos\frac{n\pi}{2} \right)^2.$$

For $E_1 = \hbar^2\pi^2/(2ma^2)$ in particular,

$$P(E_1) = \frac{4}{\pi^2} \approx 0.405.$$