Problem 2.9

For the wave function in Example 2.2, find the expectation value of H, at time $t = 0$, the "old fashioned" way:

$$\langle H \rangle = \int \Psi(x, 0)^* \hat{H} \Psi(x, 0)\, dx.$$

Compare the result we got in Example 2.3. *Note:* Because $\langle H \rangle$ is independent of time, there is no loss of generality in using $t = 0$.

Solution

In Example 2.2 the initial wave function of a particle in an infinite square well was found to be

$$\Psi(x, 0) = \sqrt{\frac{30}{a^5}} x(a - x), \quad 0 \le x \le a$$

and zero elsewhere. Use it to calculate the expectation value of the energy at $t = 0$.

$$\begin{aligned}
\langle H \rangle &= \int_{-\infty}^{\infty} \Psi^*(x, 0) \hat{H} \Psi(x, 0)\, dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x, 0) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi(x, 0)\, dx \\
&= \int_0^a \left[\sqrt{\frac{30}{a^5}} x(a - x) \right]^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 0 \right) \left[\sqrt{\frac{30}{a^5}} x(a - x) \right] dx \\
&= -\frac{\hbar^2}{2m} \int_0^a \left[\sqrt{\frac{30}{a^5}} x(a - x) \right] \frac{\partial^2}{\partial x^2} \left[\sqrt{\frac{30}{a^5}} x(a - x) \right] dx \\
&= -\frac{\hbar^2}{2m} \int_0^a \left[\sqrt{\frac{30}{a^5}} x(a - x) \right] \left[\sqrt{\frac{30}{a^5}} (-2) \right] dx \\
&= -\frac{\hbar^2}{m} \left(\frac{30}{a^5} \right) \int_0^a (x^2 - ax)\, dx \\
&= -\frac{\hbar^2}{m} \left(\frac{30}{a^5} \right) \left(-\frac{a^3}{6} \right) \\
&= \frac{5\hbar^2}{ma^2}
\end{aligned}$$

This is exactly the result found in Example 2.3, which used the alternative formula,

$$\langle H \rangle = \sum_{n=1}^{\infty} P(E_n) E_n.$$