Problem 3.11

Find the momentum-space wave function, $\Phi(p,t)$, for a particle in the ground state of the harmonic oscillator. What is the probability (to two significant digits) that a measurement of p on a particle in this state would yield a value outside the classical range (for the same energy)? *Hint:* Look in a math table under "Normal Distribution" or "Error Function" for the numerical part—or use Mathematica.

Solution

The general formulas for the Fourier transform of a function $f(x)$ and its corresponding inverse Fourier transform are as follows.

$$\begin{cases} F(k) = \sqrt{\dfrac{|b|}{(2\pi)^{1-a}}} \displaystyle\int_{-\infty}^{\infty} e^{ibkx} f(x)\,dx \\ f(x) = \sqrt{\dfrac{|b|}{(2\pi)^{1+a}}} \displaystyle\int_{-\infty}^{\infty} e^{-ibkx} F(k)\,dk \end{cases}$$

The Fourier transform can be used to solve linear partial differential equations over the whole line. Any choice for a and b is acceptable, and how one chooses to define the Fourier transform really comes down to personal preference. In Chapter 2, for example, the Schrödinger equation was solved using $a = 0$ and $b = -1$.

$$\begin{cases} \mathcal{F}\{\Psi(x,t)\} = \tilde{\Psi}(k,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} \Psi(x,t)\,dx \\ \mathcal{F}^{-1}\{\tilde{\Psi}(k,t)\} = \Psi(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{ikx} \tilde{\Psi}(k,t)\,dk \end{cases}$$

One choice for a and b is special in quantum mechanics, though: $a = 0$ and $b = -1/\hbar$.

$$\begin{cases} \mathscr{F}\{\Psi(x,t)\} = \Phi(p,t) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\,dx \\ \mathscr{F}^{-1}\{\Phi(p,t)\} = \Psi(x,t) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p,t)\,dp \end{cases}$$

$\Psi(x,t)$ is the position-space wave function because $|\Psi(x,t)|^2$ represents the probability distribution for the particle's position. On the other hand, $\Phi(p,t)$ is the momentum-space wave function because $|\Phi(p,t)|^2$ represents the probability distribution for the particle's momentum. These formulas are a result of solving the eigenvalue problem for the momentum operator.

$$\hat{p}f(x) = pf(x)$$

$$-i\hbar\frac{d}{dx}f(x) = pf(x)$$

$$\frac{df}{dx} = \frac{ip}{\hbar}f(x)$$

$$f(x) = Ae^{ipx/\hbar}$$

This is a non-normalizable function, so the spectrum is continuous, meaning the continuous Dirac-analogs of Equations 3.10 and 3.11 on page 93 apply. Since $\hat{p}$ is a hermitian operator, the eigenfunctions associated with the real, distinct eigenvalues are orthogonal.

$$\begin{aligned}\langle f' \mid f \rangle = \int_{-\infty}^{\infty} (Ae^{ip'x/\hbar})^*(Ae^{ipx/\hbar})\,dx &= \int_{-\infty}^{\infty} (A^* e^{-ip'x/\hbar})(Ae^{ipx/\hbar})\,dx \\ &= |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\,dx \\ &= A^2 \left[2\pi\delta\left(\frac{p-p'}{\hbar}\right)\right] \\ &= A^2 \left[2\pi|-\hbar|\delta(p'-p)\right] \\ &= 2\pi\hbar A^2 \delta(p'-p)\end{aligned}$$

Determine A by requiring the magnitude of the delta function to be 1.

$$2\pi\hbar A^2 = 1 \quad \rightarrow \quad A = \frac{1}{\sqrt{2\pi\hbar}}$$

Consequently,

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

$\hat{p}$ is a hermitian operator, so any function in position-space, including the one we're most interested in, $\Psi(x,t)$, can be expressed as a linear combination of its eigenfunctions.

$$\Psi(x,t) = \int_{-\infty}^{\infty} B(p,t) \left(\frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\right) dp$$

By comparing this to the general formulas, we see that this is a very special inverse Fourier transform, one where $a = 0$ and $b = -1/\hbar$. The position-space wave function for a particle in the ground state of the harmonic oscillator potential is (see Problem 2.10)

$$\Psi(x,t) = \psi_0(x) e^{-iE_0 t/\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-i\omega t/2}.$$

Take the Fourier transform of $\Psi(x,t)$ in order to get the momentum-space wave function.

$$\begin{aligned}\Phi(p,t) &= \mathscr{F}\{\Psi(x,t)\} \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\,dx \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-i\omega t/2}\,dx \\ &= \frac{e^{-i\omega t/2}}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{2\hbar}x^2 - \frac{ip}{\hbar}x\right) dx\end{aligned}$$

Complete the square in the exponent.

$$\begin{aligned}\Phi(p,t) &= \frac{e^{-i\omega t/2}}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{2ip}{m\omega}x\right)\right]dx \\ &= \frac{e^{-i\omega t/2}}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{2ip}{m\omega}x-\frac{p^2}{m^2\omega^2}\right)\right]\exp\left[-\frac{m\omega}{2\hbar}\left(\frac{p^2}{m^2\omega^2}\right)\right]dx \\ &= \frac{e^{-i\omega t/2}}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{p^2}{2\hbar m\omega}\right)\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega}{2\hbar}\left(x+\frac{ip}{m\omega}\right)^2\right]dx\end{aligned}$$

Make the following substitution.

$$u = \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)$$

$$du = \sqrt{\frac{m\omega}{2\hbar}}\,dx \quad \rightarrow \quad dx = \sqrt{\frac{2\hbar}{m\omega}}\,du$$

As a result,

$$\begin{aligned}\Phi(p,t) &= \frac{e^{-i\omega t/2}}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{p^2}{2\hbar m\omega}\right)\int_{-\infty}^{\infty}e^{-u^2}\sqrt{\frac{2\hbar}{m\omega}}\,du \\ &= \frac{e^{-i\omega t/2}}{\sqrt{\pi}}\frac{1}{(\pi\hbar m\omega)^{1/4}}\exp\left(-\frac{p^2}{2\hbar m\omega}\right)\underbrace{\int_{-\infty}^{\infty}e^{-u^2}\,du}_{=\sqrt{\pi}}\end{aligned}$$

$$\boxed{\Phi(p,t) = \frac{e^{-i\omega t/2}}{(\pi\hbar m\omega)^{1/4}}\exp\left(-\frac{p^2}{2\hbar m\omega}\right).}$$

A classical particle with mass m and energy $E \geq 0$ has momentum

$$E \geq \frac{p^2}{2m}$$

$$p^2 \leq 2mE$$

$$|p| \leq \sqrt{2mE}$$

$$-\sqrt{2mE} \leq p \leq \sqrt{2mE}.$$

Assuming it has the energy of the harmonic-oscillator ground state, $E = E_0 = \hbar\omega/2$,

$$-\sqrt{\hbar m\omega} \leq p \leq \sqrt{\hbar m\omega}.$$

The probability of measuring p outside of this range is

$$P = \int_{-\infty}^{-\sqrt{\hbar m\omega}}|\Phi(p,t)|^2\,dp + \int_{\sqrt{\hbar m\omega}}^{\infty}|\Phi(p,t)|^2\,dp$$

because $|\Phi(p,t)|^2$ is the probability distribution for the particle's momentum.

$$\begin{aligned} P &= \int_{-\infty}^{-\sqrt{\hbar m\omega}} \left| \frac{e^{-i\omega t/2}}{(\pi\hbar m\omega)^{1/4}} \exp\left(-\frac{p^2}{2\hbar m\omega}\right)\right|^2 dp + \int_{\sqrt{\hbar m\omega}}^{\infty} \left| \frac{e^{-i\omega t/2}}{(\pi\hbar m\omega)^{1/4}} \exp\left(-\frac{p^2}{2\hbar m\omega}\right)\right|^2 dp \\ &= \int_{-\infty}^{-\sqrt{\hbar m\omega}} \frac{1}{\sqrt{\pi\hbar m\omega}} \exp\left(-\frac{p^2}{\hbar m\omega}\right) dp + \int_{\sqrt{\hbar m\omega}}^{\infty} \frac{1}{\sqrt{\pi\hbar m\omega}} \exp\left(-\frac{p^2}{\hbar m\omega}\right) dp \\ &= \frac{1}{\sqrt{\pi\hbar m\omega}} \left[\int_{-\infty}^{-\sqrt{\hbar m\omega}} \exp\left(-\frac{p^2}{\hbar m\omega}\right) dp + \int_{\sqrt{\hbar m\omega}}^{\infty} \exp\left(-\frac{p^2}{\hbar m\omega}\right) dp\right] \end{aligned}$$

Make the following substitutions.

$$v = -\frac{p}{\sqrt{\hbar m\omega}} \qquad\qquad w = \frac{p}{\sqrt{\hbar m\omega}}$$

$$dv = -\frac{dp}{\sqrt{\hbar m\omega}} \quad \rightarrow \quad dp = -\sqrt{\hbar m\omega}\, dv \qquad\qquad dw = \frac{dp}{\sqrt{\hbar m\omega}} \quad \rightarrow \quad dp = \sqrt{\hbar m\omega}\, dw$$

As a result,

$$\begin{aligned} P &= \frac{1}{\sqrt{\pi\hbar m\omega}} \left[\int_{\infty}^{1} e^{-v^2} \left(-\sqrt{\hbar m\omega}\, dv\right) + \int_{1}^{\infty} e^{-w^2} \left(\sqrt{\hbar m\omega}\, dw\right)\right] \\ &= \frac{1}{\sqrt{\pi}} \left(\int_1^\infty e^{-v^2}\, dv + \int_1^\infty e^{-w^2}\, dw\right) \\ &= \frac{2}{\sqrt{\pi}} \int_1^\infty e^{-v^2}\, dv \\ &\approx \frac{2}{\sqrt{\pi}}(0.139403) \\ &\approx 0.16. \end{aligned}$$