# Problem 3.13

Show that

$$\langle x \rangle = \int \Phi^* \left( i\hbar \frac{\partial}{\partial p} \right) \Phi \, dp. \tag{3.57}$$

*Hint:* Notice that $x \exp(ipx/\hbar) = -i\hbar \, (\partial/\partial p) \exp(ipx/\hbar)$, and use Equation 2.147. In momentum space, then, the position operator is $i\hbar \, \partial/\partial p$. More generally,

$$\langle Q(x,p,t) \rangle = \begin{cases} \int \Psi^* \hat{Q} \left( x, -i\hbar \frac{\partial}{\partial x}, t \right) \Psi \, dx, & \text{in position space;} \\ \int \Phi^* \hat{Q} \left( i\hbar \frac{\partial}{\partial p}, p, t \right) \Phi \, dp, & \text{in momentum space.} \end{cases} \tag{3.58}$$

In principle you can do all calculations in momentum space just as well (though not always as *easily*) as in position space.

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### Solution

Calculate the expectation value of position at time $t$ and write it in terms of the momentum-space wave function $\Phi(p,t)$.

$$\begin{aligned}
\langle x \rangle &= \frac{\int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) \, dx}{\int_{-\infty}^{\infty} \Psi^*(x,t) \Psi(x,t) \, dx} \\
&= \frac{\int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) \, dx}{1} \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) \, dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ip'x/\hbar} \Phi(p',t) \, dp' \right]^* x \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p,t) \, dp \right] dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'x/\hbar} \Phi^*(p',t) \, dp' \right] \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} x e^{ipx/\hbar} \Phi(p,t) \, dp \right] dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'x/\hbar} \Phi^*(p',t) \, dp' \right] \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left( \frac{\hbar}{i} \frac{\partial}{\partial p} e^{ipx/\hbar} \right) \Phi(p,t) \, dp \right] dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'x/\hbar} \Phi^*(p',t) \, dp' \right] \frac{\hbar}{i} \frac{1}{\sqrt{2\pi\hbar}} \left[ \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial p} e^{ipx/\hbar} \right) \Phi(p,t) \, dp \right] dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'x/\hbar} \Phi^*(p',t) \, dp' \right] \frac{\hbar}{i} \frac{1}{\sqrt{2\pi\hbar}} \left[ \underbrace{e^{ipx/\hbar} \Phi(p,t) \Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} e^{ipx/\hbar} \frac{\partial \Phi}{\partial p} \, dp \right] dx \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'x/\hbar} \Phi^*(p',t) \, dp' \right] (i\hbar) \left[ \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \frac{\partial \Phi}{\partial p} \, dp \right] dx
\end{aligned}$$

Since the integrals have constant limits, they can be ordered in any way.

$$\begin{aligned}
\langle x\rangle &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ip'x/\hbar}\Phi^*(p',t)(i\hbar)e^{ipx/\hbar}\frac{\partial\Phi}{\partial p}\,dp'\,dp\,dx \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\,dx\right]\Phi^*(p',t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp'\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[2\pi\delta\left(\frac{p-p'}{\hbar}\right)\right]\Phi^*(p',t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp'\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[2\pi|-\hbar|\delta(p'-p)\Big]\Phi^*(p',t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp'\,dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(2\pi\hbar)\delta(p'-p)\Phi^*(p',t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp'\right]dp \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}(2\pi\hbar)\Phi^*(p,t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp \\
&= \int_{-\infty}^{\infty}\Phi^*(p,t)(i\hbar)\frac{\partial\Phi}{\partial p}\,dp
\end{aligned}$$

Therefore,

$$\langle x\rangle = \int_{-\infty}^{\infty}\Phi^*(p,t)\left(i\hbar\frac{\partial}{\partial p}\right)\Phi(p,t)\,dp.$$