Problem 3.14

(a) Prove the following commutator identities:

$$\left[\hat{A}+\hat{B},\hat{C}\right]=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right], \tag{3.64}$$

$$\left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}. \tag{3.65}$$

(b) Show that

$$[x^n,\hat{p}]=i\hbar n x^{n-1}.$$

(c) Show more generally that

$$[f(x),\hat{p}]=i\hbar\frac{df}{dx}, \tag{3.66}$$

for any function $f(x)$ that admits a Taylor series expansion.

(d) Show that for the simple harmonic oscillator

$$\left[\hat{H},\hat{a}_\pm\right]=\pm\hbar\omega\hat{a}_\pm. \tag{3.67}$$

Hint: Use Equation 2.54.

Solution

In order to prove these commutator identities, use the test function $F(x)$.

Part (a)

Prove Equation 3.64 first.

$$\begin{aligned}\left[\hat{A}+\hat{B},\hat{C}\right]F(x)&=\left[(\hat{A}+\hat{B})\hat{C}-\hat{C}(\hat{A}+\hat{B})\right]F(x)\\&=\left(\hat{A}\hat{C}+\hat{B}\hat{C}-\hat{C}\hat{A}-\hat{C}\hat{B}\right)F(x)\\&=\left[(\hat{A}\hat{C}-\hat{C}\hat{A})+(\hat{B}\hat{C}-\hat{C}\hat{B})\right]F(x)\\&=\left\{\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]\right\}F(x)\end{aligned}$$

Therefore,

$$\left[\hat{A}+\hat{B},\hat{C}\right]=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right].$$

Now prove Equation 3.65.

$$\begin{aligned}\left\{\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}\right\}F(x) &= \left[\hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B})+(\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B}\right]F(x)\\ &= \left(\hat{A}\hat{B}\hat{C}-\cancel{\hat{A}\hat{C}\hat{B}}+\cancel{\hat{A}\hat{C}\hat{B}}-\hat{C}\hat{A}\hat{B}\right)F(x)\\ &= \left[(\hat{A}\hat{B})\hat{C}-\hat{C}(\hat{A}\hat{B})\right]F(x)\\ &= \left[\hat{A}\hat{B},\hat{C}\right]F(x)\end{aligned}$$

Therefore,

$$\left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}.$$

Part (b)

$$\begin{aligned}[x^n,\hat{p}]F(x) &= \left[x^n,-i\hbar\frac{d}{dx}\right]F(x)\\ &= \left[x^n\left(-i\hbar\frac{d}{dx}\right)-\left(-i\hbar\frac{d}{dx}\right)x^n\right]F(x)\\ &= x^n\left(-i\hbar\frac{d}{dx}\right)F(x)-\left(-i\hbar\frac{d}{dx}\right)x^nF(x)\\ &= -i\hbar x^n\frac{d}{dx}[F(x)]+i\hbar\frac{d}{dx}[x^nF(x)]\\ &= -i\hbar x^n\frac{dF}{dx}+i\hbar\left[nx^{n-1}F(x)+x^n\frac{dF}{dx}\right]\\ &= \cancel{-i\hbar x^n\frac{dF}{dx}}+i\hbar nx^{n-1}F(x)+\cancel{i\hbar x^n\frac{dF}{dx}}\\ &= i\hbar nx^{n-1}F(x)\end{aligned}$$

Therefore,

$$[x^n,\hat{p}]=i\hbar nx^{n-1}.$$

Part (c)

A function $f(x)$ that has a Taylor series expansion is infinitely differentiable, so df/dx exists.

$$\begin{aligned}
[f(x), \hat{p}] &= \left[f(x), -i\hbar\frac{d}{dx}\right] F(x) \\
&= \left[f(x)\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right) f(x)\right] F(x) \\
&= f(x)\left(-i\hbar\frac{d}{dx}\right) F(x) - \left(-i\hbar\frac{d}{dx}\right) f(x)F(x) \\
&= -i\hbar f(x)\frac{d}{dx}[F(x)] + i\hbar\frac{d}{dx}[f(x)F(x)] \\
&= -i\hbar f(x)\frac{dF}{dx} + i\hbar\left[\frac{df}{dx}F(x) + f(x)\frac{dF}{dx}\right] \\
&= \cancel{-i\hbar f(x)\frac{dF}{dx}} + i\hbar\frac{df}{dx}F(x) + \cancel{i\hbar f(x)\frac{dF}{dx}} \\
&= i\hbar\frac{df}{dx}F(x)
\end{aligned}$$

Therefore,

$$[f(x), \hat{p}] = i\hbar\frac{df}{dx}.$$

Part (d)

Evaluate the commutator for the simple harmonic oscillator, using Equation 2.54 for the Hamiltonian operator.

$$\begin{aligned}
\left[\hat{H}, \hat{a}_\pm\right] F(x) &= \left[\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right), \hat{a}_\pm\right] F(x) \\
&= \left[\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_\pm - \hat{a}_\pm\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\right] F(x) \\
&= \left[\hbar\omega\hat{a}_-\hat{a}_+(\hat{a}_\pm) - \cancel{\frac{1}{2}\hbar\omega(\hat{a}_\pm)} - \hbar\omega\hat{a}_\pm(\hat{a}_-\hat{a}_+) + \cancel{\frac{1}{2}\hbar\omega(\hat{a}_\pm)}\right] F(x) \\
&= \hbar\omega\left[\hat{a}_-\hat{a}_+(\hat{a}_\pm) - \hat{a}_\pm(\hat{a}_-\hat{a}_+)\right] F(x)
\end{aligned}$$

Consider the plus and minus signs separately now and use the fact that
$[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$.

$$\begin{aligned}
\left[\hat{H}, \hat{a}_-\right] F(x) &= \hbar\omega\left[\hat{a}_-\hat{a}_+(\hat{a}_-) - \hat{a}_-(\hat{a}_-\hat{a}_+)\right] F(x) \\
&= \hbar\omega\hat{a}_-(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)F(x) \\
&= \hbar\omega\hat{a}_-(-1)F(x) \\
&= -\hbar\omega\hat{a}_-F(x)
\end{aligned}$$

With the plus sign, $\hat{a}_+$ factors to the right.

$$\begin{aligned}\left[\hat{H},\hat{a}_+\right]F(x) &= \hbar\omega\left[\hat{a}_-\hat{a}_+(\hat{a}_+) - \hat{a}_+(\hat{a}_-\hat{a}_+)\right]F(x)\\ &= \hbar\omega(\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-)\hat{a}_+F(x)\\ &= \hbar\omega(1)\hat{a}_+F(x)\\ &= \hbar\omega\hat{a}_+F(x)\end{aligned}$$

Therefore,

$$\left[\hat{H},\hat{a}_\pm\right] = \pm\hbar\omega\hat{a}_\pm.$$