Problem 3.15

Prove the famous "(your name) uncertainty principle," relating the uncertainty in position ($A = x$) to the uncertainty in energy $\big(B = p^2/2m + V\big)$:

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle p \rangle|.$$

For stationary states this doesn't tell you much—why not?

Solution

Begin with the generalized uncertainty principle for two observables, A and B.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right)^2$$

For $A = x$ and $B = p^2/2m + V(x)$, it becomes

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i} \left\langle \left[\hat{x}, \frac{\hat{p}^2}{2m} + V(\hat{x}) \right] \right\rangle \right)^2. \tag{1}$$

Evaluate the commutator by using the test function $f(x)$.

$$\begin{aligned}
\left[\hat{x}, \frac{\hat{p}^2}{2m} + V(\hat{x}) \right] f(x) &= \left[\hat{x} \left(\frac{\hat{p}^2}{2m} + V(\hat{x}) \right) - \left(\frac{\hat{p}^2}{2m} + V(\hat{x}) \right) \hat{x} \right] f(x) \\
&= \left[x \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) - \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) x \right] f(x) \\
&= x \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) f(x) - \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) x f(x) \\
&= -\frac{\hbar^2}{2m} x \frac{d^2}{dx^2} f(x) + \cancel{x V(x) f(x)} + \frac{\hbar^2}{2m} \frac{d^2}{dx^2} [x f(x)] - \cancel{V(x) x f(x)} \\
&= -\frac{\hbar^2}{2m} x \frac{d^2 f}{dx^2} + \frac{\hbar^2}{2m} \left(2 \frac{df}{dx} + x \frac{d^2 f}{dx^2} \right) \\
&= -\cancel{\frac{\hbar^2}{2m} x \frac{d^2 f}{dx^2}} + \frac{\hbar^2}{m} \frac{df}{dx} + \cancel{\frac{\hbar^2}{2m} x \frac{d^2 f}{dx^2}} \\
&= \frac{\hbar^2}{m} \frac{df}{dx} \\
&= \frac{i\hbar}{m} \left(-i\hbar \frac{d}{dx} \right) f(x) \\
&= \frac{i\hbar}{m} \hat{p} f(x)
\end{aligned}$$

Consequently,

$$\left[\hat{x}, \frac{\hat{p}^2}{2m} + V(\hat{x})\right] = \frac{i\hbar}{m}\hat{p},$$

and equation (1) becomes

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\left\langle \frac{i\hbar}{m}\hat{p}\right\rangle\right)^2$$

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$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{\hbar}{2m}\langle p\rangle\right)^2.$$

Take the square root of both sides.

$$\sigma_x \sigma_H \geq \left|\frac{\hbar}{2m}\langle p\rangle\right|$$

Therefore,

$$\boxed{\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|.}$$

For a stationary state, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$, which means the expectation value of momentum at time t is

$$\begin{aligned}
\langle p\rangle = \frac{d}{dt}\langle x\rangle = \frac{d}{dt}\langle \Psi \,|\, x \,|\, \Psi\rangle = \frac{d}{dt}\int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t)\, dx &= \frac{d}{dt}\int_{-\infty}^{\infty} [\psi(x)e^{iEt/\hbar}]x[\psi(x)e^{-iEt/\hbar}]\, dx \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} x[\psi(x)]^2\, dx \\
&= 0.
\end{aligned}$$

Calculate the expectation value of the total energy at time t, noting that $\hat{H}\psi = E\psi$.

$$\begin{aligned}
\langle H\rangle = \langle \Psi \,|\, \hat{H} \,|\, \Psi\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{H}\Psi(x,t)\, dx &= \int_{-\infty}^{\infty} [\psi(x)e^{iEt/\hbar}]\hat{H}[\psi(x)e^{-iEt/\hbar}]\, dx \\
&= \int_{-\infty}^{\infty} \psi(x)\hat{H}\psi(x)\, dx \\
&= \int_{-\infty}^{\infty} \psi(x)[E\psi(x)]\, dx \\
&= E\int_{-\infty}^{\infty} [\psi(x)]^2\, dx \\
&= E
\end{aligned}$$

Calculate the expectation value of the total energy squared at time t, noting that $\hat{H}\psi = E\psi$.

$$\begin{aligned}
\langle H^2 \rangle = \langle \Psi \mid \hat{H}^2 \mid \Psi \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{H}^2\Psi(x,t)\,dx &= \int_{-\infty}^{\infty} [\psi(x)e^{iEt/\hbar}]\hat{H}^2[\psi(x)e^{-iEt/\hbar}]\,dx \\
&= \int_{-\infty}^{\infty} \psi(x)\hat{H}^2\psi(x)\,dx \\
&= \int_{-\infty}^{\infty} \psi(x)\hat{H}[\hat{H}\psi(x)]\,dx \\
&= \int_{-\infty}^{\infty} \psi(x)\hat{H}[E\psi(x)]\,dx \\
&= E\int_{-\infty}^{\infty} \psi(x)[\hat{H}\psi(x)]\,dx \\
&= E\int_{-\infty}^{\infty} \psi(x)[E\psi(x)]\,dx \\
&= E^2\int_{-\infty}^{\infty} [\psi(x)]^2\,dx \\
&= E^2
\end{aligned}$$

The standard deviation of energy is then

$$\sigma_H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2} = \sqrt{E^2 - E^2} = 0.$$

The boxed result tells us nothing about stationary states because both sides are zero.