Problem 3.24

Show that if an operator $\hat{Q}$ is hermitian, then its matrix elements in any orthonormal basis satisfy $Q_{mn} = Q^*_{nm}$. That is, the corresponding matrix is equal to its transpose conjugate.

Solution

Suppose there's an operator $\hat{Q}$ and an orthonormal basis $|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle$. To find the matrix Q representing this operator with respect to this basis, operate on each of the basis vectors with $\hat{Q}$.

$$\begin{aligned}
\hat{Q}|e_1\rangle &= Q_{11}|e_1\rangle + Q_{21}|e_2\rangle + \cdots + Q_{n1}|e_n\rangle \\
\hat{Q}|e_2\rangle &= Q_{12}|e_1\rangle + Q_{22}|e_2\rangle + \cdots + Q_{n2}|e_n\rangle \\
&\vdots \\
\hat{Q}|e_n\rangle &= Q_{1n}|e_1\rangle + Q_{2n}|e_2\rangle + \cdots + Q_{nn}|e_n\rangle
\end{aligned}$$

These equations can be expressed compactly with the general formula,

$$\hat{Q}|e_j\rangle = \sum_{k=1}^{n} Q_{kj}|e_k\rangle, \quad 1 \leq j \leq n.$$

The aim is to solve for the matrix elements Q_{kj}. $|e_j\rangle$ is a ket, and $\hat{Q}|e_j\rangle$ is another ket. Take the inner product of both sides, then, with the bra $\langle e_i|$, where $1 \leq i \leq n$.

$$\begin{aligned}
\langle e_i| \cdot \hat{Q}|e_j\rangle &= \langle e_i| \cdot \sum_{k=1}^{n} Q_{kj}|e_k\rangle \\
&= \sum_{k=1}^{n} Q_{kj}\langle e_i \,|\, e_k\rangle \\
&= \sum_{k=1}^{n} Q_{kj}\delta_{ik} \\
&= Q_{ij}
\end{aligned}$$

Take the complex conjugate of both sides.

$$\left(\langle e_i| \cdot \hat{Q}|e_j\rangle\right)^* = Q^*_{ij}$$

Use the fact that $\langle \alpha \,|\, \beta\rangle^* = \langle \beta \,|\, \alpha\rangle$.

$$\langle e_j|\hat{Q}^\dagger \cdot |e_i\rangle = Q^*_{ij}$$

If $\hat{Q}$ is hermitian $\left(\hat{Q}^\dagger = \hat{Q}\right)$, then

$$\langle e_j|\hat{Q} \cdot |e_i\rangle = Q^*_{ij}$$

$$Q_{ji} = Q^*_{ij}.$$