Problem 3.26

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle, \quad |\beta\rangle = i|1\rangle + 2|3\rangle.$$

(a) Construct $\langle\alpha|$ and $\langle\beta|$ (in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$).

(b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$, and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

(c) Find all nine matrix elements of the operator $\hat{A} \equiv |\alpha\rangle\langle\beta|$, in this basis, and construct the matrix A. Is it hermitian?

Solution

With respect to the $|1\rangle, |2\rangle, |3\rangle$ basis, the 3×1 column matrices below represent the given kets.

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix} = \mathsf{a}$$

$$|\beta\rangle = i|1\rangle + 2|3\rangle = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix} = \mathsf{b}$$

Take the hermitian conjugate of these matrices to obtain the corresponding bras.

$$\mathsf{a}^\dagger = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix}^\dagger = \begin{pmatrix} -i & -2 & i \end{pmatrix} = -i\langle 1| - 2\langle 2| + i\langle 3| = \langle\alpha|$$

$$\mathsf{b}^\dagger = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}^\dagger = \begin{pmatrix} -i & 0 & 2 \end{pmatrix} = -i\langle 1| + 2\langle 3| = \langle\beta|$$

Evaluate the inner products, $\langle\alpha\,|\,\beta\rangle$ and $\langle\beta\,|\,\alpha\rangle$, using matrix multiplication.

$$\langle\alpha\,|\,\beta\rangle = \mathsf{a}^\dagger\mathsf{b} = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix}^\dagger \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -i & -2 & i \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix} = (-i)(i) + (-2)(0) + (i)(2) = 1 + 2i$$

$$\langle\beta\,|\,\alpha\rangle = \mathsf{b}^\dagger\mathsf{a} = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}^\dagger \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix} = \begin{pmatrix} -i & 0 & 2 \end{pmatrix} \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix} = (-i)(i) + (0)(-2) + (2)(-i) = 1 - 2i$$

Indeed, $\langle\beta\,|\,\alpha\rangle = 1 - 2i = (1 + 2i)^* = \langle\alpha\,|\,\beta\rangle^*$. Use matrix multiplication again to construct A, the matrix representing the operator $\hat{A} = |\alpha\rangle\langle\beta|$ with respect to the $|1\rangle, |2\rangle, |3\rangle$ basis.

$$\mathsf{A} = \mathsf{a}\mathsf{b}^\dagger = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}^\dagger = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix} \begin{pmatrix} -i & 0 & 2 \end{pmatrix} = \begin{pmatrix} (i)(-i) & (i)(0) & (i)(2) \\ (-2)(-i) & (-2)(0) & (-2)(2) \\ (-i)(-i) & (-i)(0) & (-i)(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

A is not hermitian because

$$\mathsf{A}^\dagger = \begin{pmatrix} 1 & -2i & -1 \\ 0 & 0 & 0 \\ -2i & -4 & 2i \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix} = \mathsf{A}.$$