Problem 3.27

Let $\hat{Q}$ be an operator with a complete set of orthonormal eigenvectors:

$$\hat{Q}|e_n\rangle = q_n|e_n\rangle \quad (n = 1, 2, 3, \ldots).$$

(a) Show that $\hat{Q}$ can be written in terms of its **spectral decomposition**:

$$\hat{Q} = \sum_n q_n |e_n\rangle\langle e_n|. \tag{3.103}$$

Hint: An operator is characterized by its action on all possible vectors, so what you must show is that

$$\hat{Q}|\alpha\rangle = \left\{\sum_n q_n |e_n\rangle\langle e_n|\right\}|\alpha\rangle,$$

for any vector $|\alpha\rangle$.

(b) Another way to define a function of $\hat{Q}$ is via the spectral decomposition:

$$f(\hat{Q}) = \sum_n f(q_n)|e_n\rangle\langle e_n|. \tag{3.104}$$

Show that this is equivalent to Equation 3.100 in the case of $e^{\hat{Q}}$.

Solution

Part (a)

Because the set of orthonormal eigenvectors is complete, any vector $|\alpha\rangle$ can be expanded as a linear combination of these eigenvectors.

$$|\alpha\rangle = \sum_{n=1}^{\infty} a_n |e_n\rangle$$

To solve for a_n, take the inner product of both sides with the bra $\langle e_i|$, where $1 \leq i < \infty$.

$$\langle e_i| \cdot |\alpha\rangle = \langle e_i| \cdot \sum_{n=1}^{\infty} a_n |e_n\rangle$$

$$\langle e_i \,|\, \alpha\rangle = \sum_{n=1}^{\infty} a_n \langle e_i \,|\, e_n\rangle$$

$$\langle e_i \,|\, \alpha\rangle = \sum_{n=1}^{\infty} a_n \delta_{in}$$

$$\langle e_n \,|\, \alpha\rangle = a_n$$

Following the hint, apply the linear operator $\hat{Q}$ to the vector $|\alpha\rangle$.

$$\begin{aligned}
\hat{Q}|\alpha\rangle &= \hat{Q}\sum_{n=1}^{\infty} a_n|e_n\rangle \\
&= \sum_{n=1}^{\infty} a_n\left(\hat{Q}|e_n\rangle\right) \\
&= \sum_{n=1}^{\infty} a_n(q_n|e_n\rangle) \\
&= \sum_{n=1}^{\infty} (q_n|e_n\rangle)a_n \\
&= \sum_{n=1}^{\infty} q_n|e_n\rangle\langle e_n \mid \alpha\rangle \\
&= \sum_{n=1}^{\infty} q_n|e_n\rangle\langle e_n| \cdot |\alpha\rangle \\
&= \left(\sum_{n=1}^{\infty} q_n|e_n\rangle\langle e_n|\right)|\alpha\rangle
\end{aligned}$$

Therefore,

$$\hat{Q} = \sum_{n=1}^{\infty} q_n|e_n\rangle\langle e_n|.$$

Part (b)

Formulas for the powers of $\hat{Q}$ can be obtained as well.

$$\hat{Q}^2 = \left(\sum_{n=1}^{\infty} q_n|e_n\rangle\langle e_n|\right)\left(\sum_{\ell=1}^{\infty} q_\ell|e_\ell\rangle\langle e_\ell|\right) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n q_\ell|e_n\rangle\langle e_n|\cdot|e_\ell\rangle\langle e_\ell| = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n q_\ell|e_n\rangle\delta_{n\ell}\langle e_\ell| = \sum_{n=1}^{\infty} q_n^2|e_n\rangle\langle e_n|$$

$$\hat{Q}^3 = \left(\sum_{n=1}^{\infty} q_n^2|e_n\rangle\langle e_n|\right)\left(\sum_{\ell=1}^{\infty} q_\ell|e_\ell\rangle\langle e_\ell|\right) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n^2 q_\ell|e_n\rangle\langle e_n|\cdot|e_\ell\rangle\langle e_\ell| = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n^2 q_\ell|e_n\rangle\delta_{n\ell}\langle e_\ell| = \sum_{n=1}^{\infty} q_n^3|e_n\rangle\langle e_n|$$

$$\hat{Q}^4 = \left(\sum_{n=1}^{\infty} q_n^3|e_n\rangle\langle e_n|\right)\left(\sum_{\ell=1}^{\infty} q_\ell|e_\ell\rangle\langle e_\ell|\right) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n^3 q_\ell|e_n\rangle\langle e_n|\cdot|e_\ell\rangle\langle e_\ell| = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} q_n^3 q_\ell|e_n\rangle\delta_{n\ell}\langle e_\ell| = \sum_{n=1}^{\infty} q_n^4|e_n\rangle\langle e_n|$$

$$\vdots$$

$$\hat{Q}^k = \sum_{n=1}^{\infty} q_n^k|e_n\rangle\langle e_n|$$

Now consider the spectral decomposition of the exponential function.

$$
\begin{aligned}
e^{\hat{Q}} &= \sum_{n=1}^{\infty} e^{q_n} |e_n\rangle\langle e_n| \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{q_n^k}{k!} \right) |e_n\rangle\langle e_n| \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} q_n^k |e_n\rangle\langle e_n| \right) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \hat{Q}^k \\
&= \hat{I} + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots
\end{aligned}
$$

This is the power series expansion of $e^{\hat{Q}}$ in Equation 3.100 on page 119.