Problem 3.29

Consider operators $\hat{A}$ and $\hat{B}$ that do not commute with each other $\left(\hat{C} = \left[\hat{A}, \hat{B}\right]\right)$ but do commute with their commutator: $\left[\hat{A}, \hat{C}\right] = \left[\hat{B}, \hat{C}\right] = 0$ (for instance, $\hat{x}$ and $\hat{p}$).

(a) Show that

$$\left[\hat{A}^n, \hat{B}\right] = n\hat{A}^{n-1}\hat{C}.$$

Hint: You can prove this by induction on n, using Equation 3.65.

(b) Show that

$$\left[e^{\lambda\hat{A}}, \hat{B}\right] = \lambda e^{\lambda\hat{A}}\hat{C},$$

where λ is any complex number. *Hint:* Express $e^{\lambda\hat{A}}$ as a power series.

(c) Derive the **Baker–Campbell–Hausdorff formula**:[37]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\hat{C}/2}.$$

Hint: Define the functions

$$\hat{f}(\lambda) = e^{\lambda\left(\hat{A}+\hat{B}\right)}, \qquad \hat{g}(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2}.$$

Note that these functions are equal at $\lambda = 0$, and show that they satisfy the same differential equation: $d\hat{f}/d\lambda = \left(\hat{A} + \hat{B}\right)\hat{f}$ and $d\hat{g}/d\lambda = \left(\hat{A} + \hat{B}\right)\hat{g}$. Therefore, the functions are themselves equal for all λ.[38]

Solution

Part (a)

The aim is to use the principle of mathematical induction to show that

$$\left[\hat{A}^n, \hat{B}\right] = n\hat{A}^{n-1}\left[\hat{A}, \hat{B}\right].$$

Start by checking the base case $n = 1$.

$$\begin{aligned}
\left[\hat{A}^1, \hat{B}\right] &\stackrel{?}{=} 1\hat{A}^{1-1}\left[\hat{A}, \hat{B}\right] \\
\left[\hat{A}, \hat{B}\right] &\stackrel{?}{=} \hat{A}^0\left[\hat{A}, \hat{B}\right] \\
&\stackrel{?}{=} \hat{I}\left[\hat{A}, \hat{B}\right] \\
&= \left[\hat{A}, \hat{B}\right]
\end{aligned}$$

[37] This is a special case of a more general formula that applies when $\hat{A}$ and $\hat{B}$ do *not* commute with $\hat{C}$. See, for example, Eugen Merzbacher, *Quantum Mechanics*, 3rd edn, Wiley, New York (1998), page 40.

[38] The product rule holds for differentiating operators as long as you respect their order:

$$\frac{d}{d\lambda}\left[\hat{A}(\lambda)\hat{B}(\lambda)\right] = \hat{A}'(\lambda)\hat{B}(\lambda) + \hat{A}(\lambda)\hat{B}'(\lambda). \tag{3.105}$$

Now make the inductive hypothesis,

$$\left[\hat{A}^k, \hat{B}\right] = k\hat{A}^{k-1}\left[\hat{A}, \hat{B}\right].$$

It must be shown that

$$\left[\hat{A}^{k+1}, \hat{B}\right] = (k+1)\hat{A}^{k}\left[\hat{A}, \hat{B}\right].$$

Work with the left side and use the commutator identity in Equation 3.65, $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$.

$$\begin{aligned}
\left[\hat{A}^{k+1}, \hat{B}\right] &= \left[\hat{A}\hat{A}^{k}, \hat{B}\right] \\
&= \hat{A}\left[\hat{A}^{k}, \hat{B}\right] + \left[\hat{A}, \hat{B}\right]\hat{A}^k \\
&= \hat{A}\left(k\hat{A}^{k-1}\left[\hat{A}, \hat{B}\right]\right) + \left[\hat{A}, \hat{B}\right]\hat{A}^k \\
&= k\hat{A}^{k}\left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{B}\right]\hat{A}^k \qquad (1)
\end{aligned}$$

Use induction again to prove the intermediate result,

$$\left[\hat{A}, \hat{B}\right]\hat{A}^n = \hat{A}^n\left[\hat{A}, \hat{B}\right].$$

Start by checking the base case, $n = 1$.

$$\left[\hat{A}, \hat{B}\right]\hat{A}^1 \stackrel{?}{=} \hat{A}^1\left[\hat{A}, \hat{B}\right]$$

$$\left[\hat{A}, \hat{B}\right]\hat{A} \stackrel{?}{=} \hat{A}\left[\hat{A}, \hat{B}\right]$$

$$\hat{C}\hat{A} = \hat{A}\hat{C}$$

Now make the inductive hypothesis,

$$\left[\hat{A}, \hat{B}\right]\hat{A}^k = \hat{A}^k\left[\hat{A}, \hat{B}\right].$$

It must be shown that

$$\left[\hat{A}, \hat{B}\right]\hat{A}^{k+1} = \hat{A}^{k+1}\left[\hat{A}, \hat{B}\right].$$

Work with the left side.

$$\begin{aligned}
\left[\hat{A}, \hat{B}\right]\hat{A}^{k+1} &= \left[\hat{A}, \hat{B}\right]\hat{A}\hat{A}^k \\
&= \hat{C}\hat{A}\hat{A}^k \\
&= \hat{A}\hat{C}\hat{A}^k \\
&= \hat{A}\left[\hat{A}, \hat{B}\right]\hat{A}^k \\
&= \hat{A}\left(\hat{A}^k\left[\hat{A}, \hat{B}\right]\right) \\
&= \hat{A}^{k+1}\left[\hat{A}, \hat{B}\right]
\end{aligned}$$

Therefore, by induction,

$$\left[\hat{A},\hat{B}\right]\hat{A}^n = \hat{A}^n\left[\hat{A},\hat{B}\right],$$

and equation (1) becomes

$$\begin{aligned}\left[\hat{A}^{k+1},\hat{B}\right] &= k\hat{A}^k\left[\hat{A},\hat{B}\right] + \left[\hat{A},\hat{B}\right]\hat{A}^k \\ &= k\hat{A}^k\left[\hat{A},\hat{B}\right] + \hat{A}^k\left[\hat{A},\hat{B}\right] \\ &= (k+1)\hat{A}^k\left[\hat{A},\hat{B}\right].\end{aligned}$$

Therefore, by induction,

$$\left[\hat{A}^n,\hat{B}\right] = n\hat{A}^{n-1}\left[\hat{A},\hat{B}\right].$$

Part (b)

The aim here is to prove that

$$\left[e^{\lambda\hat{A}},\hat{B}\right] = \lambda e^{\lambda\hat{A}}\left[\hat{A},\hat{B}\right].$$

Work with the left side.

$$\left[e^{\lambda\hat{A}},\hat{B}\right] = \left[\sum_{i=0}^{\infty}\frac{(\lambda\hat{A})^i}{i!},\hat{B}\right] \tag{2}$$

Use induction to prove the intermediate result,

$$\left[\sum_{i=0}^{n}\hat{M}_i,\hat{N}\right] = \sum_{i=0}^{n}\left[\hat{M}_i,\hat{N}\right].$$

Start by checking the base case, $n=0$.

$$\left[\sum_{i=0}^{0}\hat{M}_i,\hat{N}\right] \stackrel{?}{=} \sum_{i=0}^{0}\left[\hat{M}_i,\hat{N}\right]$$

$$\left[\hat{M}_0,\hat{N}\right] = \left[\hat{M}_0,\hat{N}\right]$$

Now make the inductive hypothesis,

$$\left[\sum_{i=0}^{k}\hat{M}_i,\hat{N}\right] = \sum_{i=0}^{k}\left[\hat{M}_i,\hat{N}\right].$$

It must be shown that

$$\left[\sum_{i=0}^{k+1}\hat{M}_i,\hat{N}\right] = \sum_{i=0}^{k+1}\left[\hat{M}_i,\hat{N}\right].$$

Work with the left side and use the commutator identity in Equation 3.64, $\left[\hat{A}+\hat{B},\hat{C}\right]=\left[\hat{A},\hat{C}\right]+\left[\hat{B},\hat{C}\right]$.

$$\begin{aligned}\left[\sum_{i=0}^{k+1}\hat{M}_i,\hat{N}\right]&=\left[\sum_{i=0}^{k}\hat{M}_i+\hat{M}_{k+1},\hat{N}\right]\\&=\left[\sum_{i=0}^{k}\hat{M}_i,\hat{N}\right]+\left[\hat{M}_{k+1},\hat{N}\right]\\&=\sum_{i=0}^{k}\left[\hat{M}_i,\hat{N}\right]+\left[\hat{M}_{k+1},\hat{N}\right]\\&=\sum_{i=0}^{k+1}\left[\hat{M}_i,\hat{N}\right]\end{aligned}$$

By induction, then,

$$\left[\sum_{i=0}^{n}\hat{M}_i,\hat{N}\right]=\sum_{i=0}^{n}\left[\hat{M}_i,\hat{N}\right].$$

Take the limit of both sides as $n\to\infty$.

$$\left[\sum_{i=0}^{\infty}\hat{M}_i,\hat{N}\right]=\sum_{i=0}^{\infty}\left[\hat{M}_i,\hat{N}\right]$$

As a result, equation (2) becomes

$$\begin{aligned}\left[e^{\lambda\hat{A}},\hat{B}\right]&=\left[\sum_{i=0}^{\infty}\frac{(\lambda\hat{A})^i}{i!},\hat{B}\right]\\&=\sum_{i=0}^{\infty}\left[\frac{(\lambda\hat{A})^i}{i!},\hat{B}\right]\\&=\sum_{i=0}^{\infty}\left[\frac{\lambda^i\hat{A}^i}{i!},\hat{B}\right]\\&=\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\left[\hat{A}^i,\hat{B}\right]\\&=\frac{\lambda^0}{0!}\left[\hat{A}^0,\hat{B}\right]+\sum_{i=1}^{\infty}\frac{\lambda^i}{i!}\left[\hat{A}^i,\hat{B}\right]\\&=\left[\hat{I},\hat{B}\right]+\sum_{i=1}^{\infty}\frac{\lambda^i}{i!}\left(i\hat{A}^{i-1}\left[\hat{A},\hat{B}\right]\right)\\&=\cancel{\hat{I}\hat{B}}-\cancel{\hat{B}\hat{I}}+\lambda\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}\hat{A}^{i-1}\left[\hat{A},\hat{B}\right].\end{aligned}$$

Continue the simplification by substituting $j = i - 1$.

$$\begin{aligned}\left[e^{\lambda\hat{A}}, \hat{B}\right] &= \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \hat{A}^j \left[\hat{A}, \hat{B}\right] \\ &= \lambda \sum_{j=0}^{\infty} \frac{(\lambda\hat{A})^j}{j!} \left[\hat{A}, \hat{B}\right]\end{aligned}$$

Therefore,

$$\left[e^{\lambda\hat{A}}, \hat{B}\right] = \lambda e^{\lambda\hat{A}} \left[\hat{A}, \hat{B}\right].$$

Part (c)

Define

$$\hat{f}(\lambda) = e^{\lambda\left(\hat{A}+\hat{B}\right)} \quad \text{and} \quad \hat{g}(\lambda) = e^{\lambda\hat{A}} e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2}.$$

Differentiate $\hat{f}(\lambda)$ with respect to λ.

$$\begin{aligned}\frac{d\hat{f}}{d\lambda} &= \frac{d}{d\lambda} e^{\lambda\left(\hat{A}+\hat{B}\right)} \\ &= e^{\lambda\left(\hat{A}+\hat{B}\right)} \cdot \frac{d}{d\lambda}\left[\lambda\left(\hat{A}+\hat{B}\right)\right] \\ &= e^{\lambda\left(\hat{A}+\hat{B}\right)}\left(\hat{A}+\hat{B}\right)\end{aligned}$$

An operator commutes with an exponential function of itself.

$$\begin{aligned}\frac{d\hat{f}}{d\lambda} &= \left(\hat{A}+\hat{B}\right) e^{\lambda\left(\hat{A}+\hat{B}\right)} \\ &= \left(\hat{A}+\hat{B}\right)\hat{f}\end{aligned}$$

Now differentiate $\hat{g}(\lambda)$ with respect to λ.

$$\begin{aligned}\frac{d\hat{g}}{d\lambda} &= \frac{d}{d\lambda}\left(e^{\lambda\hat{A}} e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2}\right) \\ &= \left[\frac{d}{d\lambda}(e^{\lambda\hat{A}})\right] e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}}\left[\frac{d}{d\lambda}(e^{\lambda\hat{B}})\right] e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}} e^{\lambda\hat{B}}\left[\frac{d}{d\lambda}(e^{-\lambda^2\hat{C}/2})\right] \\ &= \left[e^{\lambda\hat{A}} \cdot \frac{d}{d\lambda}(\lambda\hat{A})\right] e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}}\left[e^{\lambda\hat{B}} \cdot \frac{d}{d\lambda}(\lambda\hat{B})\right] e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}} e^{\lambda\hat{B}}\left[e^{-\lambda^2\hat{C}/2} \cdot \frac{d}{d\lambda}(-\lambda^2\hat{C}/2)\right] \\ &= \left[e^{\lambda\hat{A}} \cdot (\hat{A})\right] e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}}\left[e^{\lambda\hat{B}} \cdot (\hat{B})\right] e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}} e^{\lambda\hat{B}}\left[e^{-\lambda^2\hat{C}/2} \cdot (-\lambda\hat{C})\right]\end{aligned}$$

An operator commutes with an exponential function of itself.

$$\begin{aligned}&= \left[(\hat{A}) \cdot e^{\lambda\hat{A}}\right] e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} + e^{\lambda\hat{A}}\left[(\hat{B}) \cdot e^{\lambda\hat{B}}\right] e^{-\lambda^2\hat{C}/2} - \lambda e^{\lambda\hat{A}} e^{\lambda\hat{B}}\left[(\hat{C}) \cdot e^{-\lambda^2\hat{C}/2}\right] \\ &= \hat{A} e^{\lambda\hat{A}} e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} + \left(e^{\lambda\hat{A}}\hat{B}\right) e^{\lambda\hat{B}} e^{-\lambda^2\hat{C}/2} - \lambda e^{\lambda\hat{A}}\left(e^{\lambda\hat{B}}\hat{C}\right) e^{-\lambda^2\hat{C}/2}\end{aligned}$$

Use the result of part (b) in the second term. Since $\hat{C}$ commutes with $\hat{B}$, $\hat{C}$ commutes with $e^{\lambda\hat{B}}$.

$$\begin{aligned}
\frac{d\hat{g}}{d\lambda} &= \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2} + \left(\lambda e^{\lambda\hat{A}}\hat{C} + \hat{B}e^{\lambda\hat{A}}\right)e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2} - \lambda e^{\lambda\hat{A}}\left(\hat{C}e^{\lambda\hat{B}}\right)e^{-\lambda^2\hat{C}/2} \\
&= \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2} + \cancel{\lambda e^{\lambda\hat{A}}\hat{C}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2}} + \hat{B}e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2} - \cancel{\lambda e^{\lambda\hat{A}}\left(\hat{C}e^{\lambda\hat{B}}\right)e^{-\lambda^2\hat{C}/2}} \\
&= (\hat{A}+\hat{B})e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2} \\
&= (\hat{A}+\hat{B})\hat{g}
\end{aligned}$$

Both $\hat{f}$ and $\hat{g}$ satisfy the same ODE with the same initial condition at $\lambda = 0$, so $\hat{f}(\lambda) = \hat{g}(\lambda)$ for all λ.

$$e^{\lambda\left(\hat{A}+\hat{B}\right)} = e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda^2\hat{C}/2}$$

Therefore, setting $\lambda = 1$,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\hat{C}/2}.$$