Problem 3.41

A harmonic oscillator is in a state such that a measurement of the energy would yield either $(1/2)\hbar\omega$ or $(3/2)\hbar\omega$, with equal probability. What is the largest possible value of $\langle p\rangle$ in such a state? If it assumes this maximal value at time $t = 0$, what is $\Psi(x,t)$?

Solution

The general solution to the Schrödinger equation for a particle in the simple harmonic oscillator potential $V(x) = (1/2)m\omega^2x^2$ is

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar},$$

where $E_n = (n + 1/2)\hbar\omega$ and $|c_n|^2$ represents the probability that a measurement of the energy yields E_n. The wave function for the state in this problem is

$$\begin{aligned}\Psi(x,t) &= c_0\psi_0(x)e^{-iE_0t/\hbar} + c_1\psi_1(x)e^{-iE_1t/\hbar} \\ &= c_0\psi_0(x)e^{-i\omega t/2} + c_1\psi_1(x)e^{-3i\omega t/2}\end{aligned}$$

with

$$|c_0|^2 = \frac{1}{2} \qquad \text{and} \qquad |c_1|^2 = \frac{1}{2}$$

$$c_0 = \frac{e^{i\phi_0}}{\sqrt{2}} \qquad \text{and} \qquad c_1 = \frac{e^{i\phi_1}}{\sqrt{2}}.$$

The aim is to choose ϕ_0 and ϕ_1 so that $\langle p\rangle$ is maximum at time t.

$$\begin{aligned}\langle p\rangle &= \langle\Psi\,|\,\hat{p}\,|\,\Psi\rangle \\ &= \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx \\ &= -i\hbar\int_{-\infty}^{\infty}\left[c_0^*\psi_0^*(x)e^{i\omega t/2} + c_1^*\psi_1^*(x)e^{3i\omega t/2}\right]\frac{\partial}{\partial x}\left[c_0\psi_0(x)e^{-i\omega t/2} + c_1\psi_1(x)e^{-3i\omega t/2}\right]dx \\ &= -i\hbar\int_{-\infty}^{\infty}\left[c_0^*\psi_0(x)e^{i\omega t/2} + c_1^*\psi_1(x)e^{3i\omega t/2}\right]\left[c_0\psi_0'(x)e^{-i\omega t/2} + c_1\psi_1'(x)e^{-3i\omega t/2}\right]dx \\ &= -i\hbar\int_{-\infty}^{\infty}\left[c_0^*c_0\psi_0(x)\psi_0'(x) + c_0^*c_1\psi_0(x)\psi_1'(x)e^{-i\omega t} + c_1^*c_0\psi_1(x)\psi_0'(x)e^{i\omega t} + c_1^*c_1\psi_1(x)\psi_1'(x)\right]dx \\ &= -i\hbar\int_{-\infty}^{\infty}\left[\frac{1}{2}\psi_0(x)\psi_0'(x) + \frac{e^{i(\phi_1-\phi_0)}}{2}\psi_0(x)\psi_1'(x)e^{-i\omega t} + \frac{e^{-i(\phi_1-\phi_0)}}{2}\psi_1(x)\psi_0'(x)e^{i\omega t} + \frac{1}{2}\psi_1(x)\psi_1'(x)\right]dx \\ &= -\frac{i\hbar}{2}\left[\int_{-\infty}^{\infty}\psi_0(x)\psi_0'(x)\,dx + e^{i(\phi_1-\phi_0-\omega t)}\int_{-\infty}^{\infty}\psi_0(x)\psi_1'(x)\,dx\right. \\ &\qquad\left. + e^{-i(\phi_1-\phi_0-\omega t)}\int_{-\infty}^{\infty}\psi_1(x)\psi_0'(x)\,dx + \int_{-\infty}^{\infty}\psi_1(x)\psi_1'(x)\,dx\right]\end{aligned}$$

Evaluate the first integral.

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_0'(x)\,dx = \underbrace{\psi_0(x)\psi_0(x)\bigg|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \psi_0'(x)\psi_0(x)\,dx = -\int_{-\infty}^{\infty} \psi_0(x)\psi_0'(x)\,dx$$

Bring both terms to the left side.

$$2\int_{-\infty}^{\infty} \psi_0(x)\psi_0'(x)\,dx = 0$$

Divide both sides by 2.

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_0'(x)\,dx = 0$$

Evaluate the fourth integral.

$$\int_{-\infty}^{\infty} \psi_1(x)\psi_1'(x)\,dx = \underbrace{\psi_1(x)\psi_1(x)\bigg|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \psi_1'(x)\psi_1(x)\,dx = -\int_{-\infty}^{\infty} \psi_1(x)\psi_1'(x)\,dx$$

Bring both terms to the left side.

$$2\int_{-\infty}^{\infty} \psi_1(x)\psi_1'(x)\,dx = 0$$

Divide both sides by 2.

$$\int_{-\infty}^{\infty} \psi_1(x)\psi_1'(x)\,dx = 0$$

Evaluate the second integral.

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_1'(x)\,dx = \underbrace{\psi_0(x)\psi_1(x)\bigg|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty} \psi_0'(x)\psi_1(x)\,dx = -\int_{-\infty}^{\infty} \psi_1(x)\psi_0'(x)\,dx$$

Evaluate the third integral.

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_1(x)\psi_0'(x)\,dx &= \int_{-\infty}^{\infty} \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\frac{d}{dx}\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]dx \\
&= \int_{-\infty}^{\infty} \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{\hbar}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]dx \\
&= -\frac{m^2\omega^2}{\hbar^2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\
&= -\frac{2m^2\omega^2}{\hbar^2}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} x^2\exp\left[-\left(\frac{x}{\sqrt{\frac{\hbar}{m\omega}}}\right)^2\right]dx \\
&= -\frac{2m^2\omega^2}{\hbar^2}\sqrt{\frac{2}{\pi}}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3 = -\sqrt{\frac{m\omega}{2\hbar}}
\end{aligned}$$

As a result,

$$\begin{aligned}
\langle p \rangle &= -\frac{i\hbar}{2}\left[\int_{-\infty}^{\infty} \psi_0(x)\psi_0'(x)\,dx + e^{i(\phi_1-\phi_0-\omega t)}\int_{-\infty}^{\infty} \psi_0(x)\psi_1'(x)\,dx \right. \\
&\qquad \left. + e^{-i(\phi_1-\phi_0-\omega t)}\int_{-\infty}^{\infty} \psi_1(x)\psi_0'(x)\,dx + \int_{-\infty}^{\infty} \psi_1(x)\psi_1'(x)\,dx\right] \\
&= -\frac{i\hbar}{2}\left[0 + e^{i(\phi_1-\phi_0-\omega t)}\left(\sqrt{\frac{m\omega}{2\hbar}}\right) + e^{-i(\phi_1-\phi_0-\omega t)}\left(-\sqrt{\frac{m\omega}{2\hbar}}\right) + 0\right] \\
&= \sqrt{\frac{\hbar m\omega}{2}}\left[\frac{e^{i(\phi_1-\phi_0-\omega t)} - e^{-i(\phi_1-\phi_0-\omega t)}}{2i}\right] \\
&= \sqrt{\frac{\hbar m\omega}{2}}\sin(\phi_1-\phi_0-\omega t).
\end{aligned}$$

The largest possible value of $\langle p \rangle$ in the state $\Psi(x,t)$ is then

$$\langle p \rangle_{\text{max}} = \sqrt{\frac{\hbar m\omega}{2}}.$$

If this maximum is attained at $t = 0$, then

$$\begin{aligned}
\sin[\phi_1-\phi_0-\omega(0)] &= 1 \\
\phi_1 - \phi_0 &= \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2 \\
\phi_1 &= \frac{\pi}{2} + 2k\pi + \phi_0.
\end{aligned}$$

The wave function that has the largest value of $\langle p \rangle$ at $t = 0$ is therefore

$$\begin{aligned}
\Psi(x,t) &= c_0\psi_0(x)e^{-i\omega t/2} + c_1\psi_1(x)e^{-3i\omega t/2} \\
&= \frac{e^{i\phi_0}}{\sqrt{2}}\psi_0(x)e^{-i\omega t/2} + \frac{e^{i\phi_1}}{\sqrt{2}}\psi_1(x)e^{-3i\omega t/2} \\
&= \frac{e^{i\phi_0}}{\sqrt{2}}\psi_0(x)e^{-i\omega t/2} + \frac{e^{i\pi/2}e^{2ik\pi}e^{i\phi_0}}{\sqrt{2}}\psi_1(x)e^{-3i\omega t/2} \\
&= \frac{e^{i\phi_0}}{\sqrt{2}}\psi_0(x)e^{-i\omega t/2} + \frac{(i)(1)e^{i\phi_0}}{\sqrt{2}}\psi_1(x)e^{-3i\omega t/2} \\
&= \frac{e^{i\phi_0}}{\sqrt{2}}[\psi_0(x)e^{-i\omega t/2} + i\psi_1(x)e^{-3i\omega t/2}],
\end{aligned}$$

where ϕ_0 remains arbitrary.