

Problem 3.42

Coherent states of the harmonic oscillator. Among the stationary states of the harmonic oscillator (Equation 2.68) only $n = 0$ hits the uncertainty limit ($\sigma_x \sigma_p = \hbar/2$); in general, $\sigma_x \sigma_p = (2n + 1)\hbar/2$, as you found in Problem 2.12. But certain *linear combinations* (known as **coherent states**) also minimize the uncertainty product. They are (as it turns out) *eigenfunctions of the lowering operator*:⁴²

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

(the eigenvalue α can be any complex number).

- (a) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$ in the state $|\alpha\rangle$. *Hint:* Use the technique in Example 2.5, and remember that a_+ is the hermitian conjugate of a_- . Do *not* assume α is real.
- (b) Find σ_x and σ_p ; show that $\sigma_x \sigma_p = \hbar/2$.
- (c) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0.$$

- (d) Determine c_0 by normalizing $|\alpha\rangle$. *Answer:* $\exp(-|\alpha|^2/2)$.
- (e) Now put in the time dependence:

$$|n\rangle \rightarrow e^{-iE_n t/\hbar} |n\rangle,$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a_- , but the *eigenvalue* evolves in time:

$$\alpha(t) = e^{-i\omega t} \alpha.$$

So a coherent state *stays* coherent, and continues to minimize the uncertainty product.

- (f) Based on your answers to (a), (b), and (e), find $\langle x \rangle$ and σ_x as functions of time. It helps if you write the complex number α as

$$\alpha = C \sqrt{\frac{m\omega}{2\hbar}} e^{i\phi}$$

for real numbers C and ϕ . *Comment:* In a sense, coherent states behave quasi-classically.

- (g) Is the ground state ($|n = 0\rangle$) itself a coherent state? If so, what is the eigenvalue?

Solution

⁴²There are no normalizable eigenfunctions of the *raising* operator.

Part (a)

In Example 2.5 on page 47 both the position and momentum operators are written in terms of the promotion and demotion operators, \hat{a}_+ and \hat{a}_- , respectively.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

Calculate the expectation value of x in the state $|\alpha\rangle$, noting that $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$ and $\hat{a}_+^\dagger = \hat{a}_-$.

$$\begin{aligned} \langle x \rangle &= \langle \alpha | \hat{x} | \alpha \rangle \\ &= \langle \alpha | \cdot (\hat{x} | \alpha \rangle) \\ &= \langle \alpha | \cdot \left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) | \alpha \rangle \right] \\ &= \langle \alpha | \cdot \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}_+ | \alpha \rangle + \hat{a}_- | \alpha \rangle \right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle \alpha | \cdot (\hat{a}_+ | \alpha \rangle) + \langle \alpha | \cdot (\hat{a}_- | \alpha \rangle) \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ [(\hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger]^* + \langle \alpha | \cdot (\alpha | \alpha \rangle) \right\} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ [(\langle \alpha | \hat{a}_+^\dagger) \cdot | \alpha \rangle]^* + \alpha \langle \alpha | \alpha \rangle \right\} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ [\langle \alpha | \cdot (\hat{a}_- | \alpha \rangle)]^* + \alpha(1) \right\} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ [\langle \alpha | \cdot (\alpha | \alpha \rangle)]^* + \alpha \right\} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[(\alpha \langle \alpha | \alpha \rangle)^* + \alpha \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[(\alpha \cdot 1)^* + \alpha \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\alpha^* + \alpha) \\ &= \sqrt{\frac{\hbar}{2m\omega}} (2 \operatorname{Re} \alpha) \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \alpha \end{aligned}$$

Calculate the expectation value of x^2 in the state $|\alpha\rangle$. Recall that $[\hat{a}_-, \hat{a}_+] = \hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_- = 1$.

$$\begin{aligned}
\langle x^2 \rangle &= \langle \alpha | \hat{x}^2 | \alpha \rangle \\
&= \langle \alpha | \cdot (\hat{x}^2 | \alpha \rangle) \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ + \hat{a}_-) (\hat{a}_+ + \hat{a}_-) | \alpha \rangle \right] \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ \hat{a}_+ + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ + \hat{a}_- \hat{a}_-) | \alpha \rangle \right] \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ \hat{a}_+ + 2\hat{a}_+ \hat{a}_- + 1 + \hat{a}_- \hat{a}_-) | \alpha \rangle \right] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle + 2\hat{a}_+ (\hat{a}_- | \alpha \rangle) + | \alpha \rangle + \hat{a}_- (\hat{a}_- | \alpha \rangle) \right] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle + 2\hat{a}_+ (\alpha | \alpha \rangle) + | \alpha \rangle + \hat{a}_- (\alpha | \alpha \rangle) \right] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle + 2\alpha \hat{a}_+ | \alpha \rangle + | \alpha \rangle + \alpha (\hat{a}_- | \alpha \rangle) \right] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle + 2\alpha \hat{a}_+ | \alpha \rangle + | \alpha \rangle + \alpha (\alpha | \alpha \rangle) \right] \\
&= \frac{\hbar}{2m\omega} \left[\langle \alpha | \cdot \hat{a}_+ \hat{a}_+ | \alpha \rangle + 2\alpha \langle \alpha | \cdot \hat{a}_+ | \alpha \rangle + \langle \alpha | \alpha \rangle + \alpha^2 \langle \alpha | \alpha \rangle \right] \\
&= \frac{\hbar}{2m\omega} \left\{ \left[(\hat{a}_+ \hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger \right]^* + 2\alpha \left[(\hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger \right]^* + 1 + \alpha^2 \cdot 1 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \left[(\langle \alpha | \hat{a}_+^\dagger \hat{a}_+^\dagger) \cdot | \alpha \rangle \right]^* + 2\alpha \left[(\langle \alpha | \hat{a}_+^\dagger) \cdot | \alpha \rangle \right]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha | \cdot \hat{a}_- (\hat{a}_- | \alpha \rangle) \right]^* + 2\alpha \left[\langle \alpha | \cdot (\hat{a}_- | \alpha \rangle) \right]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha | \cdot \hat{a}_- (\alpha | \alpha \rangle) \right]^* + 2\alpha \left[\langle \alpha | \cdot (\alpha | \alpha \rangle) \right]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha | \cdot \alpha (\hat{a}_- | \alpha \rangle) \right]^* + 2\alpha \left(\alpha \langle \alpha | \alpha \rangle \right)^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha | \cdot \alpha (\alpha | \alpha \rangle) \right]^* + 2\alpha \left(\alpha \cdot 1 \right)^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left[\left(\alpha^2 \langle \alpha | \alpha \rangle \right)^* + 2\alpha \alpha^* + 1 + \alpha^2 \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
 \langle x^2 \rangle &= \frac{\hbar}{2m\omega} \left[(\alpha^2 \cdot 1)^* + 2\alpha\alpha^* + 1 + \alpha^2 \right] \\
 &= \frac{\hbar}{2m\omega} \left[(\alpha^*)^2 + 2\alpha^*\alpha + \alpha^2 + 1 \right] \\
 &= \frac{\hbar}{2m\omega} \left[(\alpha^* + \alpha)^2 + 1 \right] \\
 &= \frac{\hbar}{2m\omega} \left[(2 \operatorname{Re} \alpha)^2 + 1 \right] \\
 &= \frac{\hbar}{2m\omega} \left[4(\operatorname{Re} \alpha)^2 + 1 \right].
 \end{aligned}$$

Calculate the expectation value of p in the state $|\alpha\rangle$.

$$\begin{aligned}
 \langle p \rangle &= \langle \alpha | \hat{p} | \alpha \rangle \\
 &= \langle \alpha | \cdot (\hat{p} | \alpha \rangle) \\
 &= \langle \alpha | \cdot \left[i\sqrt{\frac{\hbar m \omega}{2}} (\hat{a}_+ - \hat{a}_-) | \alpha \rangle \right] \\
 &= \langle \alpha | \cdot i\sqrt{\frac{\hbar m \omega}{2}} \left(\hat{a}_+ | \alpha \rangle - \hat{a}_- | \alpha \rangle \right) \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left[\langle \alpha | \cdot (\hat{a}_+ | \alpha \rangle) - \langle \alpha | \cdot (\hat{a}_- | \alpha \rangle) \right] \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left\{ [(\hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger]^* - \langle \alpha | \cdot (\alpha | \alpha) \right\} \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left\{ [(\langle \alpha | \hat{a}_+^\dagger) \cdot | \alpha \rangle]^* - \alpha \langle \alpha | \alpha \rangle \right\} \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left\{ \left[\langle \alpha | \cdot (\hat{a}_- | \alpha) \right]^* - \alpha(1) \right\} \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left\{ \left[\langle \alpha | \cdot (\alpha | \alpha) \right]^* - \alpha \right\} \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left[(\alpha \langle \alpha | \alpha) \right]^* - \alpha \right] \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} \left[(\alpha \cdot 1)^* - \alpha \right] \\
 &= i\sqrt{\frac{\hbar m \omega}{2}} (\alpha^* - \alpha)
 \end{aligned}$$

Therefore,

$$\begin{aligned}\langle p \rangle &= 2\sqrt{\frac{\hbar m \omega}{2}} \left(\frac{\alpha - \alpha^*}{2i} \right) \\ &= \sqrt{2\hbar m \omega} \operatorname{Im} \alpha.\end{aligned}$$

Calculate the expectation value of p^2 in the state $|\alpha\rangle$.

$$\begin{aligned}\langle p^2 \rangle &= \langle \alpha | \hat{p}^2 | \alpha \rangle \\ &= \langle \alpha | \cdot (\hat{p}^2 | \alpha \rangle) \\ &= \langle \alpha | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ - \hat{a}_-) (\hat{a}_+ - \hat{a}_-) | \alpha \rangle \right] \\ &= \langle \alpha | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ \hat{a}_+ - \hat{a}_+ \hat{a}_- - \hat{a}_- \hat{a}_+ + \hat{a}_- \hat{a}_-) | \alpha \rangle \right] \\ &= \langle \alpha | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ \hat{a}_+ - 2\hat{a}_+ \hat{a}_- - 1 + \hat{a}_- \hat{a}_-) | \alpha \rangle \right] \\ &= \langle \alpha | \cdot \frac{-\hbar m \omega}{2} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle - 2\hat{a}_+ (\hat{a}_- | \alpha \rangle) - | \alpha \rangle + \hat{a}_- (\hat{a}_- | \alpha \rangle) \right] \\ &= \langle \alpha | \cdot \frac{-\hbar m \omega}{2} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle - 2\hat{a}_+ (\alpha | \alpha \rangle) - | \alpha \rangle + \hat{a}_- (\alpha | \alpha \rangle) \right] \\ &= \langle \alpha | \cdot \frac{-\hbar m \omega}{2} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle - 2\alpha \hat{a}_+ | \alpha \rangle - | \alpha \rangle + \alpha (\hat{a}_- | \alpha \rangle) \right] \\ &= \langle \alpha | \cdot \frac{-\hbar m \omega}{2} \left[\hat{a}_+ \hat{a}_+ | \alpha \rangle - 2\alpha \hat{a}_+ | \alpha \rangle - | \alpha \rangle + \alpha (\alpha | \alpha \rangle) \right] \\ &= -\frac{\hbar m \omega}{2} \left[\langle \alpha | \cdot (\hat{a}_+ \hat{a}_+ | \alpha \rangle) - 2\alpha \langle \alpha | \cdot (\hat{a}_+ | \alpha \rangle) - \langle \alpha | \alpha \rangle + \alpha^2 \langle \alpha | \alpha \rangle \right] \\ &= -\frac{\hbar m \omega}{2} \left\{ \left[(\hat{a}_+ \hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger \right]^* - 2\alpha \left[(\hat{a}_+ | \alpha \rangle)^\dagger \cdot (\langle \alpha |)^\dagger \right]^* - 1 + \alpha^2 \cdot 1 \right\} \\ &= -\frac{\hbar m \omega}{2} \left\{ \left[\left(\langle \alpha | \hat{a}_+^\dagger \hat{a}_+^\dagger \right) \cdot | \alpha \rangle \right]^* - 2\alpha \left[\left(\langle \alpha | \hat{a}_+^\dagger \right) \cdot | \alpha \rangle \right]^* - 1 + \alpha^2 \right\} \\ &= -\frac{\hbar m \omega}{2} \left\{ \left[\langle \alpha | \cdot \hat{a}_- (\hat{a}_- | \alpha \rangle) \right]^* - 2\alpha \left[\langle \alpha | \cdot (\hat{a}_- | \alpha \rangle) \right]^* - 1 + \alpha^2 \right\} \\ &= -\frac{\hbar m \omega}{2} \left\{ \left[\langle \alpha | \cdot \hat{a}_- (\alpha | \alpha \rangle) \right]^* - 2\alpha \left[\langle \alpha | \cdot (\alpha | \alpha \rangle) \right]^* - 1 + \alpha^2 \right\} \\ &= -\frac{\hbar m \omega}{2} \left\{ \left[\langle \alpha | \cdot \alpha (\hat{a}_- | \alpha \rangle) \right]^* - 2\alpha \left(\alpha | \alpha \rangle \right)^* - 1 + \alpha^2 \right\}\end{aligned}$$

Therefore,

$$\begin{aligned}
 \langle p^2 \rangle &= -\frac{\hbar m \omega}{2} \left\{ \left[\langle \alpha | \cdot \alpha (\alpha | \alpha) \rangle \right]^* - 2\alpha (\alpha \cdot 1)^* - 1 + \alpha^2 \right\} \\
 &= -\frac{\hbar m \omega}{2} \left[(\alpha^2 \langle \alpha | \alpha) \rangle^* - 2\alpha \alpha^* - 1 + \alpha^2 \right] \\
 &= -\frac{\hbar m \omega}{2} \left[(\alpha^2 \cdot 1)^* - 2\alpha \alpha^* - 1 + \alpha^2 \right] \\
 &= -\frac{\hbar m \omega}{2} [(\alpha^*)^2 - 2\alpha^* \alpha + \alpha^2 - 1] \\
 &= -\frac{\hbar m \omega}{2} [(\alpha^* - \alpha)^2 - 1] \\
 &= -\frac{\hbar m \omega}{2} \left[-4 \left(\frac{\alpha - \alpha^*}{2i} \right)^2 - 1 \right] \\
 &= \frac{\hbar m \omega}{2} [4(\text{Im } \alpha)^2 + 1].
 \end{aligned}$$

Part (b)

Now that all the expectation values are known, the standard deviations in position and momentum can be evaluated.

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega} [4(\text{Re } \alpha)^2 + 1] - \left(\sqrt{\frac{2\hbar}{m\omega}} \text{Re } \alpha \right)^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{\hbar m \omega}{2} [4(\text{Im } \alpha)^2 + 1] - \left(\sqrt{2\hbar m \omega} \text{Im } \alpha \right)^2} = \sqrt{\frac{\hbar m \omega}{2}}$$

Therefore, the uncertainty product for the state $|\alpha\rangle$ is

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m \omega}{2}} = \frac{\hbar}{2}.$$

Part (c)

Expand the coherent state $|\alpha\rangle$ in terms of the energy eigenstates of the harmonic oscillator.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

To solve for c_n , multiply both sides by $\langle q|$.

$$\begin{aligned} \langle q| \cdot |\alpha\rangle &= \langle q| \cdot \sum_{n=0}^{\infty} c_n |n\rangle \\ \langle q| \alpha\rangle &= \sum_{n=0}^{\infty} c_n \langle q|n\rangle \\ &= \sum_{n=0}^{\infty} c_n \delta_{qn} \end{aligned}$$

Because the eigenstates are orthonormal, every term in this infinite series vanishes except for one—the one for which $n = q$.

$$\langle n|\alpha\rangle = c_n$$

Recall that the n th eigenstate of the harmonic oscillator is obtained by applying the promotion operator to the ground state n times: $\psi_n(x) = A_n (\hat{a}_+)^n \psi_0(x)$, where A_n is a normalization constant.

$$\begin{aligned} c_n &= \langle n|\alpha\rangle \\ &= \langle n| \cdot |\alpha\rangle \\ &= (|n\rangle)^\dagger \cdot |\alpha\rangle \\ &= \left[A_n (\hat{a}_+)^n |0\rangle \right]^\dagger \cdot |\alpha\rangle \\ &= A_n^* \left[\langle 0| (\hat{a}_+^\dagger)^n \right] \cdot |\alpha\rangle \\ &= A_n^* \left[\langle 0| (\hat{a}_-)^n \right] \cdot |\alpha\rangle \\ &= A_n^* \langle 0| \cdot \left[(\hat{a}_-)^n |\alpha\rangle \right] \\ &= A_n^* \langle 0| \cdot \left(\alpha^n |\alpha\rangle \right) \\ &= A_n^* \alpha^n \langle 0|\alpha\rangle \\ &= A_n^* \alpha^n c_0 \end{aligned}$$

The aim now is to find the normalization constant A_n . Since the energy of the n th eigenstate is $E_n = (n + \frac{1}{2}) \hbar\omega$, the TISE for the harmonic oscillator becomes

$$\hat{H}|n\rangle = E_n|n\rangle \Rightarrow \begin{cases} \hbar\omega \left(\hat{a}_- \hat{a}_+ - \frac{1}{2} \right) |n\rangle = \left(n + \frac{1}{2} \right) \hbar\omega |n\rangle \\ \hbar\omega \left(\hat{a}_+ \hat{a}_- + \frac{1}{2} \right) |n\rangle = \left(n + \frac{1}{2} \right) \hbar\omega |n\rangle \end{cases} \rightarrow \begin{cases} \hat{a}_- \hat{a}_+ |n\rangle = (n+1)|n\rangle \\ \hat{a}_+ \hat{a}_- |n\rangle = n|n\rangle \end{cases}.$$

Suppose that the promotion and demotion operators satisfy

$$\hat{a}_+ |n\rangle = f_n |n+1\rangle \quad \text{and} \quad \hat{a}_- |n\rangle = g_n |n-1\rangle.$$

Then

$$\begin{aligned} \langle n | \hat{a}_- \rangle \cdot \langle \hat{a}_+ | n \rangle &= \langle n | \hat{a}_- \hat{a}_+ | n \rangle = \langle n | \cdot \langle \hat{a}_- \hat{a}_+ | n \rangle \\ \langle \hat{a}_-^\dagger | n \rangle \cdot \langle \hat{a}_+ | n \rangle &= \langle n | \cdot \langle (n+1) | n \rangle \\ \langle \hat{a}_+ | n \rangle \cdot \langle \hat{a}_+ | n \rangle &= (n+1) \langle n | n \rangle \\ \langle f_n | n+1 \rangle \cdot \langle f_n | n+1 \rangle &= (n+1) \cdot 1 \\ \langle f_n^* \langle n+1 | \rangle \cdot \langle f_n | n+1 \rangle &= n+1 \\ f_n^* f_n \langle n+1 | n+1 \rangle &= \\ |f_n|^2 \cdot 1 &= \\ |f_n|^2 &= \end{aligned}$$

and

$$\begin{aligned} \langle n | \hat{a}_+ \rangle \cdot \langle \hat{a}_- | n \rangle &= \langle n | \hat{a}_+ \hat{a}_- | n \rangle = \langle n | \cdot \langle \hat{a}_+ \hat{a}_- | n \rangle \\ \langle \hat{a}_+^\dagger | n \rangle \cdot \langle \hat{a}_- | n \rangle &= \langle n | \cdot \langle n | n \rangle \\ \langle \hat{a}_- | n \rangle \cdot \langle \hat{a}_- | n \rangle &= n \langle n | n \rangle \\ \langle g_n | n-1 \rangle \cdot \langle g_n | n-1 \rangle &= n \cdot 1 \\ \langle g_n^* \langle n-1 | \rangle \cdot \langle g_n | n-1 \rangle &= n, \\ g_n^* g_n \langle n-1 | n-1 \rangle &= \\ |g_n|^2 \cdot 1 &= \\ |g_n|^2 &= \end{aligned}$$

which means

$$f_n = \sqrt{n+1} \quad \text{and} \quad g_n = \sqrt{n}.$$

As a result, the promotion and demotion operators satisfy

$$\begin{aligned}\hat{a}_+|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}_-|n\rangle &= \sqrt{n}|n-1\rangle.\end{aligned}$$

Solve this first equation for the $(n+1)$ th eigenstate

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}_+|n\rangle$$

and evaluate it for several values of n to find a pattern.

$$\begin{aligned}|1\rangle &= \frac{1}{\sqrt{1}}\hat{a}_+|0\rangle \\ |2\rangle &= \frac{1}{\sqrt{2}}\hat{a}_+|1\rangle = \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{1}}\hat{a}_+|0\rangle\right) = \frac{1}{\sqrt{2\cdot 1}}(\hat{a}_+)^2|0\rangle \\ |3\rangle &= \frac{1}{\sqrt{3}}\hat{a}_+|2\rangle = \frac{1}{\sqrt{3}}\hat{a}_+\left[\frac{1}{\sqrt{2\cdot 1}}(\hat{a}_+)^2|0\rangle\right] = \frac{1}{\sqrt{3\cdot 2\cdot 1}}(\hat{a}_+)^3|0\rangle \\ |4\rangle &= \frac{1}{\sqrt{4}}\hat{a}_+|3\rangle = \frac{1}{\sqrt{4}}\hat{a}_+\left[\frac{1}{\sqrt{3\cdot 2\cdot 1}}(\hat{a}_+)^3|0\rangle\right] = \frac{1}{\sqrt{4\cdot 3\cdot 2\cdot 1}}(\hat{a}_+)^4|0\rangle \\ &\vdots \\ |n\rangle &= \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|0\rangle\end{aligned}$$

Therefore, the normalization constant is

$$A_n = \frac{1}{\sqrt{n!}},$$

and the formula for c_n becomes

$$\begin{aligned}c_n &= A_n^* \alpha^n c_0 \\ &= \frac{\alpha^n}{\sqrt{n!}} c_0.\end{aligned}$$

Part (d)

In order to determine c_0 , require that the probabilities of measuring E_0 , E_1 , and so on add to 1.

$$1 = \sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} \left| \frac{\alpha^n}{\sqrt{n!}} c_0 \right|^2 = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |c_0|^2 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} = |c_0|^2 e^{|\alpha|^2}$$

Note that $|\alpha|^2 = \alpha\alpha^*$ is a real number, so $|c_0|^2 = c_0^2$. Therefore,

$$1 = c_0^2 e^{|\alpha|^2} \quad \rightarrow \quad c_0^2 = e^{-|\alpha|^2} \quad \rightarrow \quad c_0 = e^{-|\alpha|^2/2}.$$

Part (e)

With the results of parts (c) and (d), the given eigenvalue problem for a coherent state becomes

$$\begin{aligned}\hat{a}_-|\alpha\rangle &= \alpha|\alpha\rangle \\ \hat{a}_-\left(\sum_{n=0}^{\infty} c_n|n\rangle\right) &= \alpha\left(\sum_{n=0}^{\infty} c_n|n\rangle\right) \\ \hat{a}_-\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0|n\rangle\right) &= \alpha\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0|n\rangle\right) \\ \hat{a}_-\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}|n\rangle\right) &= \alpha\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}|n\rangle\right).\end{aligned}$$

Tack on the appropriate wobble factors to the energy eigenstates to determine the coherent state's time evolution according to the Schrödinger equation.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \quad \Rightarrow \quad |\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle$$

Now apply the demotion operator to both sides.

$$\begin{aligned}\hat{a}_-|\alpha(t)\rangle &= \hat{a}_-\left(\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle\right) \\ &= \hat{a}_-\left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0 e^{-i(n+\frac{1}{2})\omega t} |n\rangle\right] \\ &= \hat{a}_-\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2} e^{-in\omega t} e^{-i\omega t/2} |n\rangle\right) \\ &= e^{-i\omega t/2} \hat{a}_-\left[\sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-|\alpha e^{-i\omega t}|^2} |n\rangle\right] \\ &= e^{-i\omega t/2} (\alpha e^{-i\omega t}) \left[\sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-|\alpha e^{-i\omega t}|^2} |n\rangle\right] \\ &= \alpha e^{-i\omega t} \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2} e^{-in\omega t} e^{-i\omega t/2} |n\rangle\right) \\ &= \alpha e^{-i\omega t} \left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0 e^{-i(n+\frac{1}{2})\omega t} |n\rangle\right] \\ &= \alpha e^{-i\omega t} \left(\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle\right) \\ &= \alpha e^{-i\omega t} |\alpha(t)\rangle\end{aligned}$$

Note that the demotion operator is in terms of position,

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}} \left(\hbar \frac{\partial}{\partial x} + m\omega x \right),$$

so the functions of time are constants as far as it's concerned. Therefore, $|\alpha(t)\rangle$ remains an eigenstate of \hat{a}_- , and the corresponding eigenvalue is $\alpha e^{-i\omega t}$.

Part (f)

Calculate the expectation value of x for the state $|\alpha(t)\rangle$.

$$\begin{aligned} \langle x \rangle &= \langle \alpha(t) | \hat{x} | \alpha(t) \rangle \\ &= \langle \alpha(t) | \cdot [\hat{x} | \alpha(t) \rangle] \\ &= \langle \alpha(t) | \cdot \left[\sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) | \alpha(t) \rangle \right] \\ &= \langle \alpha(t) | \cdot \sqrt{\frac{\hbar}{2m\omega}} \left[\hat{a}_+ | \alpha(t) \rangle + \hat{a}_- | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle \alpha(t) | \hat{a}_+ | \alpha(t) \rangle + \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle \alpha(t) | \hat{a}_+^\dagger | \alpha(t) \rangle^* + \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle^* + \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left[2 \operatorname{Re} \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \left\{ \langle \alpha(t) | \cdot [\hat{a}_- | \alpha(t) \rangle] \right\} \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \left\{ \langle \alpha(t) | \cdot [\alpha e^{-i\omega t} | \alpha(t) \rangle] \right\} \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \left[\alpha e^{-i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right] \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} (\alpha e^{-i\omega t} \cdot 1) \\ &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} (\alpha e^{-i\omega t}) \end{aligned}$$

Use the given suggestion and write the complex number α as $C\sqrt{(m\omega)/(2\hbar)}e^{i\phi}$, where C and ϕ are real numbers.

$$\begin{aligned}\langle x \rangle &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} \left[\left(C\sqrt{\frac{m\omega}{2\hbar}} e^{i\phi} \right) e^{-i\omega t} \right] \\ &= C \operatorname{Re} e^{i(\phi - \omega t)} \\ &= C \cos(\phi - \omega t)\end{aligned}$$

Therefore, at time t ,

$$\langle x \rangle = C \cos(\omega t - \phi).$$

Calculate the expectation value of x^2 for the state $|\alpha(t)\rangle$.

$$\begin{aligned}\langle x^2 \rangle &= \langle \alpha(t) | \hat{x}^2 | \alpha(t) \rangle \\ &= \langle \alpha(t) | \cdot [\hat{x}^2 | \alpha(t) \rangle] \\ &= \langle \alpha(t) | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ + \hat{a}_-) (\hat{a}_+ + \hat{a}_-) | \alpha(t) \rangle \right] \\ &= \langle \alpha(t) | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ \hat{a}_+ + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ + \hat{a}_- \hat{a}_-) | \alpha(t) \rangle \right] \\ &= \langle \alpha(t) | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ \hat{a}_+ + 2\hat{a}_+ \hat{a}_- + 1 + \hat{a}_- \hat{a}_-) | \alpha(t) \rangle \right] \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle + 2\langle \alpha(t) | \hat{a}_+ \hat{a}_- | \alpha(t) \rangle + \langle \alpha(t) | \alpha(t) \rangle + \langle \alpha(t) | \hat{a}_- \hat{a}_- | \alpha(t) \rangle \right] \\ &= \frac{\hbar}{2m\omega} \left\{ \langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle + 2 \left[\langle \alpha(t) | \hat{a}_+ \right] \cdot \left[\hat{a}_- | \alpha(t) \rangle \right] + 1 + \langle \alpha(t) | \cdot \left[\hat{a}_- \hat{a}_- | \alpha(t) \rangle \right] \right\} \\ &= \frac{\hbar}{2m\omega} \left\{ \langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle + 2 \left[\langle \alpha(t) | \hat{a}_+ \right] \cdot \left[\alpha e^{-i\omega t} | \alpha(t) \rangle \right] + 1 + \langle \alpha(t) | \cdot \left[(\alpha e^{-i\omega t})^2 | \alpha(t) \rangle \right] \right\} \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle + 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_+ | \alpha(t) \rangle + 1 + \alpha^2 e^{-2i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right] \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha(t) | \hat{a}_+^\dagger \hat{a}_+^\dagger | \alpha(t) \rangle^* + 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_+^\dagger | \alpha(t) \rangle^* + 1 + \alpha^2 e^{-2i\omega t} \right] \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha(t) | \hat{a}_- \hat{a}_- | \alpha(t) \rangle^* + 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle^* + 1 + \alpha^2 e^{-2i\omega t} \right] \\ &= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha(t) | \cdot \hat{a}_- \hat{a}_- | \alpha(t) \rangle \right]^* + 2\alpha e^{-i\omega t} \left[\langle \alpha(t) | \cdot \hat{a}_- | \alpha(t) \rangle \right]^* + 1 + \alpha^2 e^{-2i\omega t} \right\} \\ &= \frac{\hbar}{2m\omega} \left\{ \left[\langle \alpha(t) | \cdot (\alpha e^{-i\omega t})^2 | \alpha(t) \rangle \right]^* + 2\alpha e^{-i\omega t} \left[\langle \alpha(t) | \cdot \alpha e^{-i\omega t} | \alpha(t) \rangle \right]^* + 1 + \alpha^2 e^{-2i\omega t} \right\} \\ &= \frac{\hbar}{2m\omega} \left\{ \left[\alpha^2 e^{-2i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right]^* + 2\alpha e^{-i\omega t} \left[\alpha e^{-i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right]^* + 1 + \alpha^2 e^{-2i\omega t} \right\}\end{aligned}$$

Use the given suggestion and write the complex number α as $C\sqrt{(m\omega)/(2\hbar)}e^{i\phi}$, where C and ϕ are real numbers.

$$\begin{aligned}
 \langle x^2 \rangle &= \frac{\hbar}{2m\omega} \left[\left(\alpha^2 e^{-2i\omega t} \cdot 1 \right)^* + 2\alpha e^{-i\omega t} \left(\alpha e^{-i\omega t} \cdot 1 \right)^* + 1 + \alpha^2 e^{-2i\omega t} \right] \\
 &= \frac{\hbar}{2m\omega} \left[(\alpha^*)^2 e^{2i\omega t} + 2\alpha e^{-i\omega t} (\alpha^* e^{i\omega t}) + \alpha^2 e^{-2i\omega t} + 1 \right] \\
 &= \frac{\hbar}{2m\omega} \left[(\alpha^* e^{i\omega t} + \alpha e^{-i\omega t})^2 + 1 \right] \\
 &= \frac{\hbar}{2m\omega} \left\{ \left[\alpha^* e^{i\omega t} + (\alpha^* e^{i\omega t})^* \right]^2 + 1 \right\} \\
 &= \frac{\hbar}{2m\omega} \left\{ \left[2 \operatorname{Re} (\alpha^* e^{i\omega t}) \right]^2 + 1 \right\} \\
 &= \frac{\hbar}{2m\omega} \left\{ 4 \left[\operatorname{Re} \left(C \sqrt{\frac{m\omega}{2\hbar}} e^{-i\phi} e^{i\omega t} \right) \right]^2 + 1 \right\} \\
 &= \frac{\hbar}{2m\omega} \left\{ 4 \left[C \sqrt{\frac{m\omega}{2\hbar}} \operatorname{Re} e^{i(\omega t - \phi)} \right]^2 + 1 \right\} \\
 &= \frac{\hbar}{2m\omega} \left\{ 4 \left[C \sqrt{\frac{m\omega}{2\hbar}} \cos(\omega t - \phi) \right]^2 + 1 \right\} \\
 &= \frac{\hbar}{2m\omega} \left[C^2 \frac{2m\omega}{\hbar} \cos^2(\omega t - \phi) + 1 \right]
 \end{aligned}$$

Therefore, at time t ,

$$\langle x^2 \rangle = C^2 \cos^2(\omega t - \phi) + \frac{\hbar}{2m\omega}.$$

Calculate the expectation value of p for the state $|\alpha(t)\rangle$. This time, write the complex number α as $-D/\sqrt{2\hbar m\omega}e^{i\theta}$, where D and θ are real numbers.

$$\begin{aligned}
 \langle p \rangle &= \langle \alpha(t) | \hat{p} | \alpha(t) \rangle \\
 &= \langle \alpha(t) | \cdot [\hat{p} | \alpha(t) \rangle] \\
 &= \langle \alpha(t) | \cdot \left[i \sqrt{\frac{\hbar m\omega}{2}} (\hat{a}_+ - \hat{a}_-) | \alpha(t) \rangle \right] \\
 &= \langle \alpha(t) | \cdot i \sqrt{\frac{\hbar m\omega}{2}} \left[\hat{a}_+ | \alpha(t) \rangle - \hat{a}_- | \alpha(t) \rangle \right] \\
 &= i \sqrt{\frac{\hbar m\omega}{2}} \left[\langle \alpha(t) | \hat{a}_+ | \alpha(t) \rangle - \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\
 &= i \sqrt{\frac{\hbar m\omega}{2}} \left[\langle \alpha(t) | \hat{a}_+^\dagger | \alpha(t) \rangle^* - \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right]
 \end{aligned}$$

Therefore, at time t ,

$$\begin{aligned}
 \langle p \rangle &= i\sqrt{\frac{\hbar m \omega}{2}} \left[\langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle^* - \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \right] \\
 &= \sqrt{2\hbar m \omega} \left[\frac{\langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle - \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle^*}{2i} \right] \\
 &= \sqrt{2\hbar m \omega} \operatorname{Im} \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle \\
 &= \sqrt{2\hbar m \omega} \operatorname{Im} \langle \alpha(t) | \alpha e^{-i\omega t} | \alpha(t) \rangle \\
 &= \sqrt{2\hbar m \omega} \operatorname{Im} \left[\alpha e^{-i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right] \\
 &= \sqrt{2\hbar m \omega} \operatorname{Im} (\alpha e^{-i\omega t}) \\
 &= \sqrt{2\hbar m \omega} \operatorname{Im} \left(\frac{-D}{\sqrt{2\hbar m \omega}} e^{i\theta} e^{-i\omega t} \right) \\
 &= -D \operatorname{Im} e^{i(\theta - \omega t)} \\
 &= -D \sin(\theta - \omega t) \\
 &= D \sin(\omega t - \theta).
 \end{aligned}$$

Calculate the expectation value of p^2 for the state $|\alpha(t)\rangle$.

$$\begin{aligned}
 \langle p^2 \rangle &= \langle \alpha(t) | \hat{p}^2 | \alpha(t) \rangle \\
 &= \langle \alpha(t) | \cdot [\hat{p}^2 | \alpha(t) \rangle] \\
 &= \langle \alpha(t) | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ - \hat{a}_-) (\hat{a}_+ - \hat{a}_-) | \alpha(t) \rangle \right] \\
 &= \langle \alpha(t) | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ \hat{a}_+ - \hat{a}_+ \hat{a}_- - \hat{a}_- \hat{a}_+ + \hat{a}_- \hat{a}_-) | \alpha(t) \rangle \right] \\
 &= \langle \alpha(t) | \cdot \left[-\frac{\hbar m \omega}{2} (\hat{a}_+ \hat{a}_+ - 2\hat{a}_+ \hat{a}_- - 1 + \hat{a}_- \hat{a}_-) | \alpha(t) \rangle \right] \\
 &= \langle \alpha(t) | \cdot \frac{-\hbar m \omega}{2} \left[\hat{a}_+ \hat{a}_+ | \alpha(t) \rangle - 2\hat{a}_+ \hat{a}_- | \alpha(t) \rangle - | \alpha(t) \rangle + \hat{a}_- \hat{a}_- | \alpha(t) \rangle \right] \\
 &= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle - 2\langle \alpha(t) | \hat{a}_+ \hat{a}_- | \alpha(t) \rangle - \langle \alpha(t) | \alpha(t) \rangle + \langle \alpha(t) | \hat{a}_- \hat{a}_- | \alpha(t) \rangle \right] \\
 &= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle - 2\langle \alpha(t) | \hat{a}_+ (\alpha e^{-i\omega t}) | \alpha(t) \rangle - 1 + \langle \alpha(t) | (\alpha e^{-i\omega t})^2 | \alpha(t) \rangle \right] \\
 &= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | \hat{a}_+ \hat{a}_+ | \alpha(t) \rangle - 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_+ | \alpha(t) \rangle - 1 + \alpha^2 e^{-2i\omega t} \langle \alpha(t) | \alpha(t) \rangle \right]
 \end{aligned}$$

Therefore, at time t ,

$$\begin{aligned}
\langle p^2 \rangle &= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | \hat{a}_+^\dagger \hat{a}_+^\dagger | \alpha(t) \rangle^* - 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_+^\dagger | \alpha(t) \rangle^* - 1 + \alpha^2 e^{-2i\omega t} \right] \\
&= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | \hat{a}_- \hat{a}_- | \alpha(t) \rangle^* - 2\alpha e^{-i\omega t} \langle \alpha(t) | \hat{a}_- | \alpha(t) \rangle^* - 1 + \alpha^2 e^{-2i\omega t} \right] \\
&= -\frac{\hbar m \omega}{2} \left[\langle \alpha(t) | (\alpha e^{-i\omega t})^2 | \alpha(t) \rangle^* - 2\alpha e^{-i\omega t} \langle \alpha(t) | \alpha e^{-i\omega t} | \alpha(t) \rangle^* - 1 + \alpha^2 e^{-2i\omega t} \right] \\
&= -\frac{\hbar m \omega}{2} \left[(\alpha^*)^2 e^{2i\omega t} \langle \alpha(t) | \alpha(t) \rangle - 2\alpha e^{-i\omega t} \alpha^* e^{i\omega t} \langle \alpha(t) | \alpha(t) \rangle - 1 + \alpha^2 e^{-2i\omega t} \right] \\
&= -\frac{\hbar m \omega}{2} \left[(\alpha^*)^2 e^{2i\omega t} - 2\alpha \alpha^* + \alpha^2 e^{-2i\omega t} - 1 \right] \\
&= -\frac{\hbar m \omega}{2} \left[(\alpha^* e^{i\omega t} - \alpha e^{-i\omega t})^2 - 1 \right] \\
&= -\frac{\hbar m \omega}{2} \left\{ [\alpha^* e^{i\omega t} - (\alpha^* e^{i\omega t})^*]^2 - 1 \right\} \\
&= -\frac{\hbar m \omega}{2} \left\{ [2i \operatorname{Im}(\alpha^* e^{i\omega t})]^2 - 1 \right\} \\
&= -\frac{\hbar m \omega}{2} \left\{ -4 \left[\operatorname{Im} \left(\frac{-D}{\sqrt{2\hbar m \omega}} e^{-i\theta} e^{i\omega t} \right) \right]^2 - 1 \right\} \\
&= \frac{\hbar m \omega}{2} \left\{ 4 \left[\frac{-D}{\sqrt{2\hbar m \omega}} \operatorname{Im} e^{i(\omega t - \theta)} \right]^2 + 1 \right\} \\
&= \frac{\hbar m \omega}{2} \left\{ 4 \left[\frac{D^2}{2\hbar m \omega} \sin^2(\omega t - \theta) \right] + 1 \right\} \\
&= D^2 \sin^2(\omega t - \theta) + \frac{\hbar m \omega}{2}.
\end{aligned}$$

Now that all the expectation values are known, the standard deviations in position and momentum can be evaluated at time t .

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\left[C^2 \cos^2(\omega t - \phi) + \frac{\hbar}{2m\omega} \right] - [C \cos(\omega t - \phi)]^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\left[D^2 \sin^2(\omega t - \theta) + \frac{\hbar m \omega}{2} \right] - [D \sin(\omega t - \theta)]^2} = \sqrt{\frac{\hbar m \omega}{2}}$$

The uncertainty product for the state $|\alpha(t)\rangle$ at time t is

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m \omega}{2}} = \frac{\hbar}{2}.$$

Therefore, a coherent state stays coherent and continues to minimize the uncertainty product.

Part (g)

The ground state of the harmonic oscillator is a coherent state with eigenvalue 0 because

$$\begin{aligned}
 \hat{a}_-|0\rangle &= \frac{1}{\sqrt{2\hbar m\omega}} \left(\hbar \frac{d}{dx} + m\omega x \right) \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\
 &= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}} \left[\hbar \frac{d}{dx} \left(e^{-\frac{m\omega}{2\hbar}x^2} \right) + m\omega x \left(e^{-\frac{m\omega}{2\hbar}x^2} \right) \right] \\
 &= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}} \left[\hbar \left(e^{-\frac{m\omega}{2\hbar}x^2} \right) \frac{d}{dx} \left(-\frac{m\omega}{2\hbar}x^2 \right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2} \right] \\
 &= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}} \left[\hbar \left(e^{-\frac{m\omega}{2\hbar}x^2} \right) \left(-\frac{m\omega}{\hbar}x \right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2} \right] \\
 &= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}} \left(-m\omega x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2} \right) \\
 &= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}} (0) \\
 &= 0 \\
 &= 0 \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\
 &= 0|0\rangle.
 \end{aligned}$$