Problem 3.46

The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathsf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables, A and B, are represented by the matrices

$$\mathsf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathsf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ, and μ are positive real numbers.

(a) Find the eigenvalues and (normalized) eigenvectors of H, A, and B.

(b) Suppose the system starts out in the generic state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation values (at $t = 0$) of H, A, and B.

(c) What is $|\mathcal{S}(t)\rangle$? If you measured the energy of this state (at time t), what values might you get, and what is the probability of each? Answer the same questions for observables A and for B.

[TYPO: Remove this second instance of "for."]

Solution

The Hamiltonian Matrix

The eigenvalue problem for the Hamiltonian operator $\hat{H}$ is the TISE.

$$\hat{H}|s\rangle = E|s\rangle$$

With respect to a certain basis, the given Hamiltonian matrix H represents the Hamiltonian operator.

$$(\mathsf{H} - E\mathsf{I})|s\rangle = \mathsf{0} \tag{1}$$

Since $|s\rangle$ can't be the zero vector, the matrix in parentheses must be singular.

$$\det(\mathsf{H} - E\mathsf{I}) = 0$$

$$\begin{vmatrix} \hbar\omega - E & 0 & 0 \\ 0 & 2\hbar\omega - E & 0 \\ 0 & 0 & 2\hbar\omega - E \end{vmatrix} = 0$$

$$(\hbar\omega - E) \begin{vmatrix} 2\hbar\omega - E & 0 \\ 0 & 2\hbar\omega - E \end{vmatrix} = 0$$

Evaluate the determinant and solve for the eigenvalues, the allowed energies.

$$(\hbar\omega - E)(2\hbar\omega - E)^2 = 0$$

$$E = \{\hbar\omega, 2\hbar\omega\}$$

Even though there are only two eigenvalues, H is hermitian (and normal as a result), so H is diagonalizable and its eigenvectors form a basis for the three-dimensional space. Notice that H is already diagonalized, with the eigenvalues along the main diagonal. Let $E_- = \hbar\omega$ and $E_+ = 2\hbar\omega$ and plug each of them into equation (1) to determine the associated eigenvectors.

$$(\mathsf{H} - E_-\mathsf{I})|s_-\rangle = \mathsf{0} \qquad\qquad (\mathsf{H} - E_+\mathsf{I})|s_+\rangle = \mathsf{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar\omega & 0 \\ 0 & 0 & \hbar\omega \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} -\hbar\omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \hbar\omega s_2 &= 0 \\ \hbar\omega s_3 &= 0 \end{aligned}\right\} \qquad\qquad -\hbar\omega s_1 = 0$$

$$\left.\begin{aligned} s_2 &= 0 \\ s_3 &= 0 \end{aligned}\right\} \qquad\qquad s_1 = 0$$

Consequently,

$$|s_-\rangle = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ 0 \\ 0 \end{pmatrix} = s_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |s_+\rangle = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ s_2 \\ s_3 \end{pmatrix} = s_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The three basis vectors are then

$$|s_-\rangle = s_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |s_{+1}\rangle = s_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |s_{+2}\rangle = s_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For them to be physically relevant, they have to be normalized.

$$s_1^2 = 1 \qquad\qquad s_2^2 = 1 \qquad\qquad s_3^2 = 1$$

$$s_1 = \pm 1 \qquad\qquad s_2 = \pm 1 \qquad\qquad s_3 = \pm 1$$

Therefore, the normalized eigenvectors of H are

$$|s_-\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |s_{+1}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |s_{+2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Suppose that the system has the initial state vector,

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then the expectation value of H at $t = 0$ is

$$\langle H\rangle = \frac{\langle \mathcal{S}(0)\,|\,\hat{H}\,|\,\mathcal{S}(0)\rangle}{\langle \mathcal{S}(0)\,|\,\mathcal{S}(0)\rangle} = \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \mathsf{H} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}} = \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{c_1^*c_1 + c_2^*c_2 + c_3^*c_3}$$

$$= \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} \hbar\omega c_1 \\ 2\hbar\omega c_2 \\ 2\hbar\omega c_3 \end{pmatrix}}{|c_1|^2 + |c_2|^2 + |c_3|^2}$$

$$= \frac{c_1^*\hbar\omega c_1 + 2c_2^*\hbar\omega c_2 + 2c_3^*\hbar\omega c_3}{1}$$

$$= \hbar\omega(c_1^*c_1 + 2c_2^*c_2 + 2c_3^*c_3)$$

$$= \hbar\omega(|c_1|^2 + 2|c_2|^2 + 2|c_3|^2).$$

Now write the initial state vector as a linear combination of the normalized eigenvectors.

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = A_1|s_-\rangle + B_1|s_{+1}\rangle + C_1|s_{+2}\rangle$$

$$= A_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + B_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix}$$

As a result, $A_1 = c_1$ and $B_1 = c_2$ and $C_1 = c_3$, which means

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1|s_-\rangle + c_2|s_{+1}\rangle + c_3|s_{+2}\rangle.$$

In order to get the state vector at any later time t, multiply each term by the corresponding wiggle factor.

$$|\mathcal{S}(t)\rangle = c_1|s_-\rangle e^{-iE_-t/\hbar} + c_2|s_{+1}\rangle e^{-iE_+t/\hbar} + c_3|s_{+2}\rangle e^{-iE_+t/\hbar}$$

$$= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i(\hbar\omega)t/\hbar} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-i(2\hbar\omega)t/\hbar} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-i(2\hbar\omega)t/\hbar} = \begin{pmatrix} c_1e^{-i\omega t} \\ c_2e^{-2i\omega t} \\ c_3e^{-2i\omega t} \end{pmatrix}$$

The probability of measuring $E_- = \hbar\omega$ is $|c_1|^2$, and the probability of measuring $E_+ = 2\hbar\omega$ is $|c_2|^2 + |c_3|^2$.

Observable A

Solve the eigenvalue problem for the operator representing observable A now.

$$\hat{A}|\alpha\rangle = a|\alpha\rangle$$

With respect to a certain basis, the given matrix A represents the operator $\hat{A}$.

$$(\mathsf{A} - a\mathsf{I})|\alpha\rangle = \mathsf{0} \tag{2}$$

Since $|\alpha\rangle$ can't be the zero vector, the matrix in parentheses must be singular.

$$\det(\mathsf{A} - a\mathsf{I}) = 0$$

$$\begin{vmatrix} -a & \lambda & 0 \\ \lambda & -a & 0 \\ 0 & 0 & 2\lambda - a \end{vmatrix} = 0$$

$$(2\lambda - a)\begin{vmatrix} -a & \lambda \\ \lambda & -a \end{vmatrix} = 0$$

Evaluate the determinant and solve for the eigenvalues, the possible measurements of A.

$$(2\lambda - a)(a^2 - \lambda^2) = 0$$

$$(2\lambda - a)(a + \lambda)(a - \lambda) = 0$$

$$a = \{-\lambda, \lambda, 2\lambda\}$$

Let $a_- = -\lambda$ and $a_0 = \lambda$ and $a_+ = 2\lambda$.

Plug each of them into equation (2) to determine the associated eigenvectors.

$$(\mathsf{A} - a_-\mathsf{I})|\alpha_-\rangle = 0 \qquad (\mathsf{A} - a_0\mathsf{I})|\alpha_0\rangle = 0 \qquad (\mathsf{A} - a_+\mathsf{I})|\alpha_+\rangle = 0$$

$$\begin{pmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 3\lambda \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -\lambda & \lambda & 0 \\ \lambda & -\lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -2\lambda & \lambda & 0 \\ \lambda & -2\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \lambda\alpha_1 + \lambda\alpha_2 &= 0 \\ 3\lambda\alpha_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -\lambda\alpha_1 + \lambda\alpha_2 &= 0 \\ \lambda\alpha_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -2\lambda\alpha_1 + \lambda\alpha_2 &= 0 \\ \lambda\alpha_1 - 2\lambda\alpha_2 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha_2 &= -\alpha_1 \\ \alpha_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha_1 &= \alpha_2 \\ \alpha_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \end{aligned}\right\}$$

Consequently,

$$|\alpha_-\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\alpha_1 \\ 0 \end{pmatrix} = \alpha_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad |\alpha_0\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \alpha_2 \\ 0 \end{pmatrix} = \alpha_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad |\alpha_+\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix} = \alpha_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For these eigenvectors to be physically relevant, they must be normalized.

$$(\alpha_1)^2 + (-\alpha_1)^2 + (0)^2 = 1 \qquad (\alpha_2)^2 + (\alpha_2)^2 + (0)^2 = 1 \qquad (0)^2 + (0)^2 + (\alpha_3)^2 = 1$$

$$\alpha_1 = \pm\frac{1}{\sqrt{2}} \qquad \alpha_2 = \pm\frac{1}{\sqrt{2}} \qquad \alpha_3 = \pm 1$$

Therefore, the normalized eigenvectors of A are

$$|\alpha_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\alpha_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\alpha_+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If the system has the initial state vector,

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

then the expectation value of A at $t = 0$ is

$$\begin{aligned}
\langle A \rangle = \frac{\langle \mathcal{S}(0) \,|\, \hat{A} \,|\, \mathcal{S}(0) \rangle}{\langle \mathcal{S}(0) \,|\, \mathcal{S}(0) \rangle} &= \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \mathsf{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}} = \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{c_1^* c_1 + c_2^* c_2 + c_3^* c_3} \\
&= \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} \lambda c_2 \\ \lambda c_1 \\ 2\lambda c_3 \end{pmatrix}}{|c_1|^2 + |c_2|^2 + |c_3|^2} \\
&= \frac{c_1^* \lambda c_2 + c_2^* \lambda c_1 + 2 c_3^* \lambda c_3}{1} \\
&= \lambda(c_1^* c_2 + c_2^* c_1 + 2 c_3^* c_3) \\
&= \lambda(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2).
\end{aligned}$$

If A is measured at time t, the eigenvalues, $a_- = -\lambda$ and $a_0 = \lambda$ and $a_+ = 2\lambda$, are the values that you can get. To find the probabilities of getting them, start with the basic formula relating them with the expectation value of A and use projection operators like so.

$$\begin{aligned}
\sum_n a_n P(a_n) = \langle A \rangle = \frac{\langle \mathcal{S}(t) \,|\, \hat{A} \,|\, \mathcal{S}(t) \rangle}{\langle \mathcal{S}(t) \,|\, \mathcal{S}(t) \rangle} &= \frac{\langle \mathcal{S}(t) \,|\, \hat{I} \hat{A} \hat{I} \,|\, \mathcal{S}(t) \rangle}{\begin{pmatrix} c_1^* e^{i\omega t} & c_2^* e^{2i\omega t} & c_3^* e^{2i\omega t} \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix}} \\
&= \frac{\left\langle \mathcal{S}(t) \middle| \left(\sum_m |\alpha_m\rangle\langle\alpha_m| \right) \mathsf{A} \left(\sum_n |\alpha_n\rangle\langle\alpha_n| \right) \middle| \mathcal{S}(t) \right\rangle}{c_1^* c_1 + c_2^* c_2 + c_3^* c_3} \\
&= \frac{\sum_m \sum_n \langle \mathcal{S}(t) \,|\, \alpha_m \rangle \langle \alpha_m | \cdot (\mathsf{A} |\alpha_n\rangle) \, \langle \alpha_n \,|\, \mathcal{S}(t) \rangle}{|c_1|^2 + |c_2|^2 + |c_3|^2} \\
&= \frac{\sum_m \sum_n \langle \mathcal{S}(t) \,|\, \alpha_m \rangle \langle \alpha_m | \cdot (a_n |\alpha_n\rangle) \, \langle \alpha_n \,|\, \mathcal{S}(t) \rangle}{1}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\sum_n a_n P(a_n) &= \sum_m \sum_n a_n \langle \mathcal{S}(t) \,|\, \alpha_m \rangle \langle \alpha_m \,|\, \alpha_n \rangle \langle \alpha_n \,|\, \mathcal{S}(t) \rangle \\
&= \sum_m \sum_n a_n \langle \mathcal{S}(t) \,|\, \alpha_m \rangle \delta_{mn} \langle \alpha_n \,|\, \mathcal{S}(t) \rangle \\
&= \sum_n a_n \langle \mathcal{S}(t) \,|\, \alpha_n \rangle \langle \alpha_n \,|\, \mathcal{S}(t) \rangle \\
&= \sum_n a_n \langle \alpha_n \,|\, \mathcal{S}(t) \rangle^* \langle \alpha_n \,|\, \mathcal{S}(t) \rangle \\
&= \sum_n a_n |\langle \alpha_n \,|\, \mathcal{S}(t) \rangle|^2
\end{aligned}$$

Consequently, the probabilities of measuring $a_- = -\lambda$ and $a_0 = \lambda$ and $a_+ = 2\lambda$ are respectively

$$\begin{aligned}
P(a_-) &= |\langle \alpha_- \,|\, \mathcal{S}(t) \rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\
&= \frac{1}{2} \left| c_1 e^{-i\omega t} - c_2 e^{-2i\omega t} \right|^2 \\
&= \frac{1}{2} (c_1^* e^{i\omega t} - c_2^* e^{2i\omega t})(c_1 e^{-i\omega t} - c_2 e^{-2i\omega t}) \\
&= \frac{1}{2} (c_1^* c_1 - c_1^* c_2 e^{-i\omega t} - c_2^* c_1 e^{i\omega t} + c_2^* c_2) \\
&= \frac{1}{2} (|c_1|^2 - c_1^* c_2 e^{-i\omega t} - c_2^* c_1 e^{i\omega t} + |c_2|^2)
\end{aligned}$$

$$\begin{aligned}
P(a_0) &= |\langle \alpha_0 \,|\, \mathcal{S}(t) \rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\
&= \frac{1}{2} \left| c_1 e^{-i\omega t} + c_2 e^{-2i\omega t} \right|^2 \\
&= \frac{1}{2} (c_1^* e^{i\omega t} + c_2^* e^{2i\omega t})(c_1 e^{-i\omega t} + c_2 e^{-2i\omega t}) \\
&= \frac{1}{2} (c_1^* c_1 + c_1^* c_2 e^{-i\omega t} + c_2^* c_1 e^{i\omega t} + c_2^* c_2) \\
&= \frac{1}{2} (|c_1|^2 + c_1^* c_2 e^{-i\omega t} + c_2^* c_1 e^{i\omega t} + |c_2|^2)
\end{aligned}$$

$$\begin{aligned}
P(a_+) &= |\langle \alpha_+ \,|\, \mathcal{S}(t) \rangle|^2 \\
&= \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\
&= \left| c_3 e^{-2i\omega t} \right|^2 \\
&= (c_3^* e^{2i\omega t})(c_3 e^{-2i\omega t}) \\
&= c_3^* c_3 \\
&= |c_3|^2.
\end{aligned}$$

Observable B

Solve the eigenvalue problem for the operator representing observable B now.

$$\hat{B}|\beta\rangle = b|\beta\rangle$$

With respect to a certain basis, the given matrix B represents the operator $\hat{B}$.

$$(\mathsf{B} - b\mathsf{I})|\beta\rangle = \mathsf{0} \tag{3}$$

Since $|\beta\rangle$ can't be the zero vector, the matrix in parentheses must be singular.

$$\det(\mathsf{B} - b\mathsf{I}) = 0$$

$$\begin{vmatrix} 2\mu - b & 0 & 0 \\ 0 & -b & \mu \\ 0 & \mu & -b \end{vmatrix} = 0$$

$$(2\mu - b)\begin{vmatrix} -b & \mu \\ \mu & -b \end{vmatrix} = 0$$

Evaluate the determinant and solve for the eigenvalues, the possible measurements of B.

$$(2\mu - b)(b^2 - \mu^2) = 0$$

$$(2\mu - b)(b + \mu)(b - \mu) = 0$$

$$b = \{-\mu, \mu, 2\mu\}$$

Let $b_- = -\mu$ and $b_0 = \mu$ and $b_+ = 2\mu$.

Plug each of them into equation (3) to determine the associated eigenvectors.

$$(\mathsf{B} - b_-\mathsf{I})|\beta_-\rangle = 0 \qquad (\mathsf{B} - b_0\mathsf{I})|\beta_0\rangle = 0 \qquad (\mathsf{B} - b_+\mathsf{I})|\beta_+\rangle = 0$$

$$\begin{pmatrix} 3\mu & 0 & 0 \\ 0 & \mu & \mu \\ 0 & \mu & \mu \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \mu & 0 & 0 \\ 0 & -\mu & \mu \\ 0 & \mu & -\mu \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\mu & \mu \\ 0 & \mu & -2\mu \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 3\mu\beta_1 &= 0 \\ \mu\beta_2 + \mu\beta_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \mu\beta_1 &= 0 \\ -\mu\beta_2 + \mu\beta_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -2\mu\beta_2 + \mu\beta_3 &= 0 \\ \mu\beta_2 - 2\mu\beta_3 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \beta_1 &= 0 \\ \beta_2 &= -\beta_3 \end{aligned}\right\} \qquad \left.\begin{aligned} \beta_1 &= 0 \\ \beta_3 &= \beta_2 \end{aligned}\right\} \qquad \left.\begin{aligned} \beta_2 &= 0 \\ \beta_3 &= 0 \end{aligned}\right\}$$

Consequently,

$$|\beta_-\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta_3 \\ \beta_3 \end{pmatrix} = \beta_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad |\beta_0\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_2 \\ \beta_2 \end{pmatrix} = \beta_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad |\beta_+\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ 0 \\ 0 \end{pmatrix} = \beta_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For these eigenvectors to be physically relevant, they must be normalized.

$$(0)^2 + (-\beta_3)^2 + (\beta_3)^2 = 1 \qquad (0)^2 + (\beta_2)^2 + (\beta_2)^2 = 1 \qquad (\beta_1)^2 + (0)^2 + (0)^2 = 1$$

$$\beta_3 = \pm\frac{1}{\sqrt{2}} \qquad \beta_2 = \pm\frac{1}{\sqrt{2}} \qquad \beta_1 = \pm 1$$

Therefore, the normalized eigenvectors of B are

$$|\beta_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\beta_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\beta_+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If the system has the initial state vector,

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

then the expectation value of B at $t = 0$ is

$$\langle B\rangle = \frac{\langle \mathcal{S}(0)\,|\,\hat{B}\,|\,\mathcal{S}(0)\rangle}{\langle \mathcal{S}(0)\,|\,\mathcal{S}(0)\rangle} = \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \mathsf{B} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}} = \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}{c_1^*c_1 + c_2^*c_2 + c_3^*c_3}$$

$$= \frac{\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2\mu c_1 \\ \mu c_3 \\ \mu c_2 \end{pmatrix}}{|c_1|^2 + |c_2|^2 + |c_3|^2}$$

$$= \frac{2c_1^*\mu c_1 + c_2^*\mu c_3 + c_3^*\mu c_2}{1}$$

$$= \mu(2c_1^*c_1 + c_2^*c_3 + c_3^*c_2)$$

$$= \mu(2|c_1|^2 + c_2^*c_3 + c_3^*c_2).$$

If B is measured at time t, the eigenvalues, $b_- = -\mu$ and $b_0 = \mu$ and $b_+ = 2\mu$, are the values that you can get. To find the probabilities of getting them, start with the basic formula relating them with the expectation value of B and use projection operators like so.

$$\sum_n b_n P(b_n) = \langle B\rangle = \frac{\langle \mathcal{S}(t)\,|\,\hat{B}\,|\,\mathcal{S}(t)\rangle}{\langle \mathcal{S}(t)\,|\,\mathcal{S}(t)\rangle} = \frac{\langle \mathcal{S}(t)\,|\,\hat{I}\hat{B}\hat{I}\,|\,\mathcal{S}(t)\rangle}{\begin{pmatrix} c_1^*e^{i\omega t} & c_2^*e^{2i\omega t} & c_3^*e^{2i\omega t} \end{pmatrix} \begin{pmatrix} c_1e^{-i\omega t} \\ c_2e^{-2i\omega t} \\ c_3e^{-2i\omega t} \end{pmatrix}}$$

$$= \frac{\left\langle \mathcal{S}(t) \,\middle|\, \left(\sum_m |\beta_m\rangle\langle\beta_m|\right) \mathsf{B} \left(\sum_n |\beta_n\rangle\langle\beta_n|\right) \,\middle|\, \mathcal{S}(t) \right\rangle}{c_1^*c_1 + c_2^*c_2 + c_3^*c_3}$$

$$= \frac{\sum_m \sum_n \langle \mathcal{S}(t)\,|\,\beta_m\rangle\langle\beta_m| \cdot (\mathsf{B}|\beta_n\rangle)\,\langle\beta_n\,|\,\mathcal{S}(t)\rangle}{|c_1|^2 + |c_2|^2 + |c_3|^2}$$

$$= \frac{\sum_m \sum_n \langle \mathcal{S}(t)\,|\,\beta_m\rangle\langle\beta_m| \cdot (b_n|\beta_n\rangle)\,\langle\beta_n\,|\,\mathcal{S}(t)\rangle}{1}$$

Continue the simplification.

$$\begin{aligned}\sum_n b_n P(b_n) &= \sum_m \sum_n b_n \langle \mathcal{S}(t) \,|\, \beta_m \rangle \langle \beta_m \,|\, \beta_n \rangle \langle \beta_n \,|\, \mathcal{S}(t) \rangle \\ &= \sum_m \sum_n b_n \langle \mathcal{S}(t) \,|\, \beta_m \rangle \delta_{mn} \langle \beta_n \,|\, \mathcal{S}(t) \rangle \\ &= \sum_n b_n \langle \mathcal{S}(t) \,|\, \beta_n \rangle \langle \beta_n \,|\, \mathcal{S}(t) \rangle \\ &= \sum_n b_n \langle \beta_n \,|\, \mathcal{S}(t) \rangle^* \langle \beta_n \,|\, \mathcal{S}(t) \rangle \\ &= \sum_n b_n |\langle \beta_n \,|\, \mathcal{S}(t) \rangle|^2\end{aligned}$$

Consequently, the probabilities of measuring $b_- = -\mu$ and $b_0 = \mu$ and $b_+ = 2\mu$ are respectively

$$\begin{aligned}P(b_-) &= |\langle \beta_- \,|\, \mathcal{S}(t) \rangle|^2 \\ &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \left| -c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t} \right|^2 \\ &= \frac{1}{2} (-c_2^* e^{2i\omega t} + c_3^* e^{2i\omega t})(-c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t}) \\ &= \frac{1}{2} (c_2^* c_2 - c_2^* c_3 - c_3^* c_2 + c_3^* c_3) \\ &= \frac{1}{2} (|c_2|^2 - c_2^* c_3 - c_3^* c_2 + |c_3|^2)\end{aligned}$$

$$\begin{aligned}P(b_0) &= |\langle \beta_0 \,|\, \mathcal{S}(t) \rangle|^2 \\ &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\ &= \frac{1}{2} \left| c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t} \right|^2 \\ &= \frac{1}{2} (c_2^* e^{2i\omega t} + c_3^* e^{2i\omega t})(c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t}) \\ &= \frac{1}{2} (c_2^* c_2 + c_2^* c_3 + c_3^* c_2 + c_3^* c_3) \\ &= \frac{1}{2} (|c_2|^2 + c_2^* c_3 + c_3^* c_2 + |c_3|^2)\end{aligned}$$

$$\begin{aligned}P(b_+) &= |\langle \beta_+ \,|\, \mathcal{S}(t) \rangle|^2 \\ &= \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{-i\omega t} \\ c_2 e^{-2i\omega t} \\ c_3 e^{-2i\omega t} \end{pmatrix} \right|^2 \\ &= \left| c_1 e^{-i\omega t} \right|^2 \\ &= (c_1^* e^{i\omega t})(c_1 e^{-i\omega t}) \\ &= c_1^* c_1 \\ &= |c_1|^2.\end{aligned}$$