Problem 3.5

(a) Find the hermitian conjugates of x, i, and d/dx.

(b) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$ (note the reversed order), $(\hat{Q}+\hat{R}) = \hat{Q}^\dagger + \hat{R}^\dagger$, and $(c\hat{Q})^\dagger = c^*\hat{Q}^\dagger$ for a complex number c.

(c) Construct the hermitian conjugate of a_+ (Equation 2.48).

[TYPO: This should be $(\hat{Q}+\hat{R})^\dagger$.]

Solution

The hermitian conjugate of an operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that

$$\langle f \,|\, \hat{Q}g\rangle = \langle \hat{Q}^\dagger f \,|\, g\rangle$$

for all $f(x)$ and $g(x)$. Find the hermitian conjugate of x, a real number.

$$\begin{aligned}
\langle f \,|\, xg\rangle &= \langle f \,|\, x \,|\, g\rangle \\
&= \int_{-\infty}^{\infty} f^*(x) x g(x)\, dx \\
&= \int_{-\infty}^{\infty} x f^*(x) g(x)\, dx \\
&= \int_{-\infty}^{\infty} x^* f^*(x) g(x)\, dx \\
&= \int_{-\infty}^{\infty} [x f(x)]^* g(x)\, dx \\
&= \langle x f \,|\, g\rangle
\end{aligned}$$

Therefore, $x^\dagger = x$. Now find the hermitian conjugate of $i = \sqrt{-1}$.

$$\begin{aligned}
\langle f \,|\, ig\rangle &= \langle f \,|\, i \,|\, g\rangle \\
&= \int_{-\infty}^{\infty} f^*(x) i g(x)\, dx \\
&= \int_{-\infty}^{\infty} i f^*(x) g(x)\, dx \\
&= \int_{-\infty}^{\infty} (-i)^* f^*(x) g(x)\, dx \\
&= \int_{-\infty}^{\infty} [-i f(x)]^* g(x)\, dx \\
&= \langle -i f \,|\, g\rangle
\end{aligned}$$

Therefore, $i^\dagger = -i$.

Now find the hermitian conjugate of d/dx, the derivative with respect to x. Note that $f(x)$ and $g(x)$ are complex-valued functions, so $f(x)$ can be written as $u(x) + iv(x)$, where $u(x)$ and $v(x)$ are real-valued functions. Additionally, $f(x)$ and $g(x)$ tend to zero as $x \to \pm\infty$.

$$\begin{aligned}
\left\langle f \,\middle|\, \frac{d}{dx} g \right\rangle &= \left\langle f \,\middle|\, \frac{d}{dx} \,\middle|\, g \right\rangle \\
&= \int_{-\infty}^{\infty} f^*(x) \frac{d}{dx} g(x)\, dx \\
&= \int_{-\infty}^{\infty} f^*(x) \frac{dg}{dx}\, dx \\
&= \underbrace{f^*(x) g(x) \bigg|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \frac{df^*}{dx} g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \frac{df^*}{dx} g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \frac{d}{dx} [u(x) + iv(x)]^* g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \frac{d}{dx} [u(x) - iv(x)] g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \left(\frac{du}{dx} - i \frac{dv}{dx} \right) g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \left(\frac{du}{dx} + i \frac{dv}{dx} \right)^* g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \left\{ \frac{d}{dx} [u(x) + iv(x)] \right\}^* g(x)\, dx \\
&= -\int_{-\infty}^{\infty} \left(\frac{df}{dx} \right)^* g(x)\, dx \\
&= \int_{-\infty}^{\infty} (-1)^* \left(\frac{df}{dx} \right)^* g(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(-\frac{df}{dx} \right)^* g(x)\, dx \\
&= \left\langle -\frac{d}{dx} f \,\middle|\, g \right\rangle
\end{aligned}$$

Therefore,

$$\left(\frac{d}{dx} \right)^\dagger = -\frac{d}{dx}.$$

Now find the hermitian conjugate of $\hat{Q}\hat{R}$, the product of two operators.

$$\begin{aligned}
\langle f \,|\, \hat{Q}\hat{R}g\rangle &= \langle f \,|\, \hat{Q}\hat{R} \,|\, g\rangle \\
&= \int_{-\infty}^{\infty} f^*(x)\hat{Q}\hat{R}g(x)\,dx \\
&= \int_{-\infty}^{\infty} f^*(x)\left\{\hat{Q}\left[\hat{R}g(x)\right]\right\}dx \\
&= \int_{-\infty}^{\infty} \left[\hat{Q}^\dagger f(x)\right]^* \left[\hat{R}g(x)\right]dx \\
&= \int_{-\infty}^{\infty} \left\{\hat{R}^\dagger\left[\hat{Q}^\dagger f(x)\right]\right\}^* g(x)\,dx \\
&= \int_{-\infty}^{\infty} \left[\hat{R}^\dagger\hat{Q}^\dagger f(x)\right]^* g(x)\,dx \\
&= \langle \hat{R}^\dagger\hat{Q}^\dagger f \,|\, g\rangle
\end{aligned}$$

Therefore, $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$. Now find the hermitian conjugate of $\hat{Q}+\hat{R}$, the sum of two operators.

$$\begin{aligned}
\langle f \,|\, (\hat{Q}+\hat{R})g\rangle &= \langle f \,|\, (\hat{Q}+\hat{R}) \,|\, g\rangle \\
&= \int_{-\infty}^{\infty} f^*(x)(\hat{Q}+\hat{R})g(x)\,dx \\
&= \int_{-\infty}^{\infty} f^*(x)[\hat{Q}g(x)+\hat{R}g(x)]\,dx \\
&= \int_{-\infty}^{\infty} [f^*(x)\hat{Q}g(x)+f^*(x)\hat{R}g(x)]\,dx \\
&= \int_{-\infty}^{\infty} f^*(x)\hat{Q}g(x)\,dx + \int_{-\infty}^{\infty} f^*(x)\hat{R}g(x)\,dx \\
&= \int_{-\infty}^{\infty} \left[\hat{Q}^\dagger f(x)\right]^* g(x)\,dx + \int_{-\infty}^{\infty} \left[\hat{R}^\dagger f(x)\right]^* g(x)\,dx \\
&= \int_{-\infty}^{\infty} \left\{\left[\hat{Q}^\dagger f(x)\right]^* g(x) + \left[\hat{R}^\dagger f(x)\right]^* g(x)\right\}dx \\
&= \int_{-\infty}^{\infty} \left\{\left[\hat{Q}^\dagger f(x)\right]^* + \left[\hat{R}^\dagger f(x)\right]^*\right\} g(x)\,dx \\
&= \int_{-\infty}^{\infty} \left[\hat{Q}^\dagger f(x) + \hat{R}^\dagger f(x)\right]^* g(x)\,dx \\
&= \int_{-\infty}^{\infty} \left[\left(\hat{Q}^\dagger + \hat{R}^\dagger\right) f(x)\right]^* g(x)\,dx \\
&= \left\langle \left(\hat{Q}^\dagger + \hat{R}^\dagger\right) f \,|\, g\right\rangle
\end{aligned}$$

Therefore, $(\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$.

Now find the hermitian conjugate of $c\hat{Q}$, where c is a complex constant.

$$\begin{aligned}
\langle f \,|\, c\hat{Q}g\rangle &= \langle f \,|\, c\hat{Q} \,|\, g\rangle \\
&= \int_{-\infty}^{\infty} f^*(x) c\hat{Q} g(x)\, dx \\
&= c\int_{-\infty}^{\infty} f^*(x) \hat{Q} g(x)\, dx \\
&= c\int_{-\infty}^{\infty} \left[\hat{Q}^\dagger f(x)\right]^* g(x)\, dx \\
&= \int_{-\infty}^{\infty} c\left[\hat{Q}^\dagger f(x)\right]^* g(x)\, dx \\
&= \int_{-\infty}^{\infty} (c^*)^*\left[\hat{Q}^\dagger f(x)\right]^* g(x)\, dx \\
&= \int_{-\infty}^{\infty} \left[c^*\hat{Q}^\dagger f(x)\right]^* g(x)\, dx \\
&= \langle c^*\hat{Q}^\dagger f \,|\, g\rangle
\end{aligned}$$

Therefore, $(c\hat{Q})^\dagger = c^*\hat{Q}^\dagger$. Now use all these proven results to find the hermitian conjugate of $\hat{a}_+$, the raising operator for the harmonic oscillator.

$$\begin{aligned}
\hat{a}_+^\dagger = \left[\frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\right]^\dagger &= \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^*(-i\hat{p} + m\omega\hat{x})^\dagger \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[(-i\hat{p})^\dagger + (m\omega\hat{x})^\dagger\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[(-i)^*\hat{p}^\dagger + (m\omega)^*\hat{x}^\dagger\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[i\left(-i\hbar\frac{d}{dx}\right)^\dagger + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[i(-i\hbar)^*\left(\frac{d}{dx}\right)^\dagger + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[i(i\hbar)\left(-\frac{d}{dx}\right) + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[i\left(-i\hbar\frac{d}{dx}\right) + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}) \\
&= \hat{a}_-
\end{aligned}$$

Finally, find the hermitian conjugate of $\hat{a}_-$, the lowering operator for the harmonic oscillator.

$$\begin{aligned}
\hat{a}_-^\dagger &= \left[\frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})\right]^\dagger \\
&= \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^* (i\hat{p} + m\omega\hat{x})^\dagger \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[(i\hat{p})^\dagger + (m\omega\hat{x})^\dagger\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[(i)^*\hat{p}^\dagger + (m\omega)^*\hat{x}^\dagger\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right)^\dagger + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[-i(-i\hbar)^*\left(\frac{d}{dx}\right)^\dagger + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[-i(i\hbar)\left(-\frac{d}{dx}\right) + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right) + m\omega\hat{x}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) \\
&= \hat{a}_+
\end{aligned}$$

Therefore, $\hat{a}_+^\dagger = \hat{a}_-$ and $\hat{a}_-^\dagger = \hat{a}_+$.