Problem 3.6

Consider the operator $\hat{Q} = d^2/d\phi^2$, where (as in Example 3.1) ϕ is the azimuthal angle in polar coordinates, and the functions are subject to Equation 3.26. Is $\hat{Q}$ hermitian? Find its eigenfunctions and eigenvalues. What is the spectrum of $\hat{Q}$? Is the spectrum degenerate?

[TYPO: Replace "polar" with "spherical."]

Solution

The operator $\hat{Q}$ is hermitian if $\langle f \,|\, \hat{Q}f\rangle = \langle \hat{Q}f \,|\, f\rangle$ for all $f(\phi)$, where $f(\phi + 2\pi) = f(\phi)$ and $0 \le \phi \le 2\pi$.

$$\begin{aligned}
\langle f \,|\, \hat{Q}f\rangle &= \langle f \,|\, \hat{Q} \,|\, f\rangle \\
&= \left\langle f \,\middle|\, \frac{d^2}{d\phi^2} \,\middle|\, f \right\rangle \\
&= \int_0^{2\pi} f^*(\phi) \frac{d^2}{d\phi^2} f(\phi)\, d\phi \\
&= \int_0^{2\pi} f^*(\phi) \frac{d^2 f}{d\phi^2}\, d\phi
\end{aligned}$$

Integrate by parts twice in order to transfer the derivatives onto the conjugated function.

$$\begin{aligned}
\langle f \,|\, \hat{Q}f\rangle &= \underbrace{f^*(\phi)\frac{df}{d\phi}\bigg|_0^{2\pi}}_{=\,0} - \int_0^{2\pi} \frac{df^*}{d\phi}\frac{df}{d\phi}\, d\phi \\
&= -\int_0^{2\pi} \frac{df^*}{d\phi}\frac{df}{d\phi}\, d\phi \\
&= -\Bigg[\underbrace{\frac{df^*}{d\phi} f(\phi)\bigg|_0^{2\pi}}_{=\,0} - \int_0^{2\pi} \frac{d^2 f^*}{d\phi^2} f(\phi)\, d\phi\Bigg] \\
&= \int_0^{2\pi} \frac{d^2 f^*}{d\phi^2} f(\phi)\, d\phi
\end{aligned}$$

$f(\phi)$ is a complex-valued function, so it can be written as $f(\phi) = u(\phi) + iv(\phi)$, where $u(\phi)$ and $v(\phi)$ are real-valued functions.

$$\begin{aligned}
\langle f \,|\, \hat{Q}f\rangle &= \int_0^{2\pi} \frac{d^2}{d\phi^2}[u(\phi) + iv(\phi)]^* f(\phi)\, d\phi \\
&= \int_0^{2\pi} \frac{d^2}{d\phi^2}[u(\phi) - iv(\phi)] f(\phi)\, d\phi \\
&= \int_0^{2\pi} \left(\frac{d^2 u}{d\phi^2} - i\frac{d^2 v}{d\phi^2}\right) f(\phi)\, d\phi
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\langle f \,|\, \hat{Q} f\rangle &= \int_0^{2\pi} \left(\frac{d^2u}{d\phi^2} + i\frac{d^2v}{d\phi^2}\right)^* f(\phi)\,d\phi \\
&= \int_0^{2\pi} \left\{\frac{d^2}{d\phi^2}[u(\phi) + iv(\phi)]\right\}^* f(\phi)\,d\phi \\
&= \int_0^{2\pi} \left[\frac{d^2}{d\phi^2} f(\phi)\right]^* f(\phi)\,d\phi \\
&= \left\langle \frac{d^2}{d\phi^2} f \,\middle|\, f \right\rangle \\
&= \langle \hat{Q} f \,|\, f\rangle
\end{aligned}$$

Therefore, $\hat{Q} = d^2/d\phi^2$ is hermitian. Now solve the eigenvalue problem for $\hat{Q}$.

$$\hat{Q} f(\phi) = q f(\phi)$$

$$\frac{d^2 f}{d\phi^2} = q f(\phi)$$

Since $0 \leq \phi \leq 2\pi$ and $f(\phi + 2\pi) = f(\phi)$, the associated boundary conditions are periodic: $f(2\pi) = f(0)$ and $f'(2\pi) = f'(0)$. Check to see if there are positive eigenvalues first: $q = \mu^2$.

$$\frac{d^2 f}{d\phi^2} = \mu^2 f(\phi)$$

The general solution to this ODE can be written in terms of hyperbolic sine and hyperbolic cosine.

$$f(\phi) = C_1 \cosh \mu\phi + C_2 \sinh \mu\phi$$

Differentiate it once with respect to ϕ.

$$f'(\phi) = \mu(C_1 \sinh \mu\phi + C_2 \cosh \mu\phi)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned}
f(0) = f(2\pi) \quad &\rightarrow \quad C_1 = C_1 \cosh 2\pi\mu + C_2 \sinh 2\pi\mu \\
f'(0) = f'(2\pi) \quad &\rightarrow \quad \mu C_2 = \mu(C_1 \sinh 2\pi\mu + C_2 \cosh 2\pi\mu)
\end{aligned}$$

Solve this first equation for C_2

$$C_2 = \frac{1 - \cosh 2\pi\mu}{\sinh 2\pi\mu} C_1$$

and plug it into the second equation.

$$\frac{1 - \cosh 2\pi\mu}{\sinh 2\pi\mu} C_1 = C_1 \sinh 2\pi\mu + \frac{1 - \cosh 2\pi\mu}{\sinh 2\pi\mu} C_1 \cosh 2\pi\mu$$

Assume $C_1 \neq 0$ and multiply both sides by $(\sinh 2\pi\mu)/C_1$.

$$\begin{aligned} 1 - \cosh 2\pi\mu &= \sinh^2 2\pi\mu + (1 - \cosh 2\pi\mu)\cosh 2\pi\mu \\ &= (\sinh^2 2\pi\mu - \cosh^2 2\pi\mu) + \cosh 2\pi\mu \\ &= -1 + \cosh 2\pi\mu \end{aligned}$$

Simplify the equation.

$$\cosh 2\pi\mu = 1$$

Since there are no nonzero values of μ that satisfy this equation, there are no positive eigenvalues. Now check to see if zero is an eigenvalue: $q = 0$.

$$\frac{d^2 f}{d\phi^2} = 0$$

The general solution is obtained by integrating both sides with respect to ϕ twice.

$$f(\phi) = C_3\phi + C_4$$

Differentiate it once with respect to ϕ.

$$f'(\phi) = C_3$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} f(0) &= f(2\pi) & &\rightarrow & C_4 &= 2\pi C_3 + C_4 \\ f'(0) &= f'(2\pi) & &\rightarrow & C_3 &= C_3 \end{aligned}$$

Solving this first equation gives $C_3 = 0$, and C_4 remains arbitrary.

$$f(\phi) = C_4$$

This is a nontrivial solution, so zero is an eigenvalue. Finally, check to see if there are negative eigenvalues: $q = -\gamma^2$.

$$\frac{d^2 f}{d\phi^2} = -\gamma^2 f(\phi)$$

The general solution to this ODE can be written in terms of sine and cosine.

$$f(\phi) = C_5 \cos\gamma\phi + C_6 \sin\gamma\phi$$

Differentiate it once with respect to ϕ.

$$f'(\phi) = \gamma(-C_5 \sin\gamma\phi + C_6 \cos\gamma\phi)$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} f(0) &= f(2\pi) & &\rightarrow & C_5 &= C_5 \cos 2\pi\gamma + C_6 \sin 2\pi\gamma \\ f'(0) &= f'(2\pi) & &\rightarrow & \gamma C_6 &= \gamma(-C_5 \sin 2\pi\gamma + C_6 \cos 2\pi\gamma) \end{aligned}$$

Solve this first equation for C_6

$$C_6 = \frac{1 - \cos 2\pi\gamma}{\sin 2\pi\gamma} C_5$$

and plug it into the second equation.

$$\frac{1-\cos 2\pi\gamma}{\sin 2\pi\gamma}C_5 = -C_5\sin 2\pi\gamma + \frac{1-\cos 2\pi\gamma}{\sin 2\pi\gamma}C_5\cos 2\pi\gamma$$

Assume $C_5 \neq 0$ and multiply both sides by $(\sin 2\pi\gamma)/C_5$.

$$\begin{aligned} 1-\cos 2\pi\gamma &= -\sin^2 2\pi\gamma + (1-\cos 2\pi\gamma)\cos 2\pi\gamma \\ &= -(\sin^2 2\pi\gamma + \cos^2 2\pi\gamma) + \cos 2\pi\gamma \\ &= -1 + \cos 2\pi\gamma \end{aligned}$$

Simplify the equation.

$$\cos 2\pi\gamma = 1$$

Solve for γ.

$$\begin{aligned} 2\pi\gamma &= 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots \\ \gamma &= n \end{aligned}$$

There are negative eigenvalues, $q = -n^2$, where $n = 1, 2, \dots$ because $n = 0$ leads to the zero eigenvalue and negative values of n lead to redundant eigenvalues. As a result,

$$f(\phi) = C_5\cos n\phi + C_6\sin n\phi.$$

The formula for C_6 doesn't apply because $\sin 2\pi\gamma = 0$. Reapply the boundary conditions, then, to determine C_5 and C_6.

$$\begin{aligned} f(0) &= f(2\pi) \quad &\rightarrow \quad C_5 &= C_5 \\ f'(0) &= f'(2\pi) \quad &\rightarrow \quad nC_6 &= nC_6 \end{aligned}$$

Both C_5 and C_6 actually remain arbitrary. In conclusion, the eigenvalues of $\hat{Q} = d^2/d\phi^2$ are

$$q = -n^2, \quad n = 0, 1, 2, \dots,$$

and the eigenfunctions associated with them are $\cos n\phi$ and $\sin n\phi$. This collection of eigenvalues (the spectrum) is degenerate because multiple linearly independent eigenfunctions are associated with an eigenvalue.