Problem 3.7

(a) Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with the same eigenvalue q. Show that any linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.

(b) Check that $f(x) = \exp(x)$ and $g(x) = \exp(-x)$ are eigenfunctions of the operator d^2/dx^2, with the same eigenvalue. Construct two linear combinations of f and g that are *orthogonal* eigenfunctions on the interval $(-1, 1)$.

Solution

Part (a)

Let $f(x)$ and $g(x)$ be two eigenfunctions of an operator $\hat{Q}$ with the same eigenvalue q.

$$\hat{Q}f(x) = qf(x)$$

$$\hat{Q}g(x) = qg(x)$$

Multiply both sides of the first equation by a, a complex constant, and multiply both sides of the second equation by b, a complex constant.

$$a\hat{Q}f(x) = aqf(x)$$

$$b\hat{Q}g(x) = bqg(x)$$

Add the respective sides of these equations.

$$a\hat{Q}f(x) + b\hat{Q}g(x) = aqf(x) + bqg(x)$$

Use the fact that $\hat{Q}$ is a linear operator on the left side.

$$\hat{Q}[af(x) + bg(x)] = aqf(x) + bqg(x)$$

Factor the right side.

$$\hat{Q}[af(x) + bg(x)] = q[af(x) + bg(x)]$$

Therefore, any linear combination of $f(x)$ and $g(x)$ is also an eigenfunction of $\hat{Q}$ with eigenvalue q.

Part (b)

Apply the operator d^2/dx^2 to both e^x and e^{-x}.

$$\frac{d^2}{dx^2}(e^x) = 1(e^x)$$

$$\frac{d^2}{dx^2}(e^{-x}) = (-1)^2(e^{-x}) = 1(e^{-x})$$

This verifies that e^x and e^{-x} are eigenfunctions of d^2/dx^2 with eigenvalue 1.

Let $F(x) = e^x + e^{-x}$ and $G(x) = e^x - e^{-x}$, for example. Then

$$\begin{aligned}
\langle F \,|\, G \rangle &= \int_{-1}^{1} (e^x + e^{-x})^* (e^x - e^{-x})\, dx \\
&= \int_{-1}^{1} (e^x + e^{-x})(e^x - e^{-x})\, dx \\
&= \int_{-1}^{1} (e^{2x} - e^{-2x})\, dx \\
&= \left(\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right) \Bigg|_{-1}^{1} \\
&= \frac{1}{2}(e^2 - e^{-2}) + \frac{1}{2}(e^{-2} - e^2) \\
&= 0,
\end{aligned}$$

which means $F(x)$ and $G(x)$ are orthogonal on the interval $-1 < x < 1$. Notice that $F(x)$ and $G(x)$ are also eigenfunctions of d^2/dx^2 with eigenvalue 1, as they are linear combinations of e^x and e^{-x}.

$$\begin{aligned}
\frac{d^2 F}{dx^2} &= \frac{d^2}{dx^2}(e^x + e^{-x}) = 1(e^x + e^{-x}) \\
\frac{d^2 G}{dx^2} &= \frac{d^2}{dx^2}(e^x - e^{-x}) = 1(e^x - e^{-x})
\end{aligned}$$